AN INTRODUCTION

TO

PROJECTIVE GEOMETRY

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PREFACE

MY object in writing the following pages has been to supply the growing need of mathematical students in this country for a compact text-book giving the theory of Conic Sections on modern lines. During recent years increasing space has been allowed, in University syllabuses and courses of instruction, to the more powerful and general projective methods, as opposed to the more special methods of what is still known as Geometrical Conics.

The line of cleavage between the two has, however, been sharply maintained, with the result that the already much overworked mathematical student has to learn his theory of Conic Sections three times over: (1) analytically; (2) according to Euclidean methods; (3) according to Projective methods.

The difficulty has been to reconcile the Euclidean and Projective definitions of the curve; in fact to bring in the focal properties into Projective Geometry at a sufficiently early stage. The practice has usually been, in order to pass from the projective to the focal definitions, to introduce the theory of involution. But the latter requires for its fullest and clearest treatment the employment of imaginary elements. It seems undesirable that the more fundamental focal properties of the conics, e.g. the sum or difference of the focal distances and the angles made by these with the tangent and normal, should appear to depend upon properties of imaginary points and lines, even though this might introduce greater rapidity of treatment. The University of London has recognized this, for, while admitting Projective Geometry into its syllabuses for the Final Examination for a Pass Degree, it has excluded involution. Many teachers have felt that this exclusion amounted to a rigorous enforcement of the line of cleavage mentioned above. In the present book the difficulty has been met, it is hoped, successfully. Chapters I—VII cover practically the whole field of Geometry of

Conics which is required from the average mathematical student who is not reading for Honours. In these chapters no use has been made of involution. In Chapter VI a proof is given that any conic, projectively defined, can be cut from a real right circular cone. The foci are then obtained from the focal spheres and the rest of the focal properties follow.

In Chapters I—VII the only knowledge presupposed is that of Euclid, Books I—VI and Book XI; also enough of Analytical Geometry to understand the use of signs and coordinates and the meaning of the equations of the straight line and circle, so that the data of the drawing examples should be intelligible to the reader. Of plane Trigonometry the meaning of the sine, cosine and tangent and the formula for the area of a triangle (area $=\frac{1}{2}ab\sin C$) are all that is required.

Instead of basing the treatment of the subject upon harmonic groups, I have introduced cross-ratio from the very beginning. Although fully appreciating the superior elegance of the former method, insomuch as it enables Projective Geometry to be developed without any appeal to metrical properties, I think it is hardly the one best suited to beginners. For this reason I have used metrical methods whenever their use was obviously indicated, although I hope it will be found that the spirit of the projective methods has been adhered to.

The second part of the book is intended for students reading for Honours or desirous of making themselves familiar with the more advanced parts of the subject. It has been impossible, in such a short space, to do more than bring the reader to the threshold of the rich treasure house of Modern Geometry and to give him a glimpse of some of its more characteristic methods. No attempt has been made to give a complete account of the theorems obtained in this domain: those given have been chosen to illustrate the methods. Nevertheless it is believed that an Honours student will find there most, if not all, of the fundamental results which he ought to know.

The range of knowledge presupposed on the part of the reader is, of course, much wider in the later than in the earlier chapters. Thus the whole theory of imaginaries and of homography has been allowed to rest on an analytical basis. This should present no difficulty, for, by the time the reader reaches these chapters, he will almost certainly have acquired sufficient knowledge of Analytical Plane and Solid Geometry to make his progress easy. On the other hand a purely geometrical development of imaginaries would have been too long and laborious for inclusion. But it will be found that those results which depend on analytical considerations are in every case broad generaliza-

tions, such as those relating to the properties of conjugate imaginaries, to the operations which lead to homographic relations and to the number of points in which curves and surfaces of given degree and order intersect. I have carefully abstained from using analysis to prove particular theorems.

With regard to homography the method of one-one correspondence has been made fundamental. It is true that discrimination has to be used in applying the principle, but this may be said of almost any principle: and a student soon gets to know when a one-one correspondence geometrically given is really algebraic. The notion of homographic involutions, which appears to be a powerful instrument, has been introduced in Chapter XI.

Finally Chapters XIII and XIV deal with geometry of space. Many properties of cones of the second order, of sphero-conics, and of quadrics come most easily from purely geometrical considerations: and it seems a pity that the methods of Pure Geometry are not more frequently employed at this stage.

In the preparation of the book the classical treatises of Cremona and Reye and a more recent but very concise and instructive exposition of the subject by M. Ernest Duporcq have been chiefly consulted.

I have ventured to make certain changes in the recognized nomenclature. Thus what is called by Mr Russell in his *Treatise of Pure Geometry* the axis and pole of homography I call the *cross-axis* and *cross-centre*, as the name seems to bring more vividly before the mind the fundamental property of the thing defined. For a similar reason I have used the term *incident* to denote two forms such that the elements of one lie in the corresponding elements of the other. The term perspective, which is employed by German writers in this connexion, appears misleading, since it would not then apply to what are universally known as perspective ranges and pencils.

The examples are taken mostly from exercises set to classes at University College, London, and from College and University Examination papers. For permission to use these my thanks are due to the Principal of the University of London and to the Provost of University College, London. The examples contain also many theorems which it has not been found possible to include in the text. A special feature of those on the first seven chapters is that they are divided into two sets. Those marked (A) are theoretical; those marked (B) are drawing exercises. My own experience as a teacher leads me to believe that such actual drawing is of immense value in assisting beginners to understand the subject, as well as intrinsically useful in practice.

Considerable stress has therefore been laid upon drawing-board constructions.

I wish to express my deep sense of obligation to my friend and colleague, Mr J. H. Dibb, B.Sc., of University College, London, for the help he has given me both before and during the passage of the book through the Press. To Mr H. J. Harris, B.A., I also owe most hearty thanks for some valuable criticisms.

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CONTENTS

PROJECTIVE GEOMETRY

CHAPTER I.

PROJECTION.

1. Corresponding figures. Geometrical properties may be obtained by correspondence or transformation. Certain relations are assumed, which introduce a correspondence between the *elements* of one figure (namely its points, lines and planes) and the elements of another figure. The two figures are then said to correspond or to be transformable into one another. To any property of a set of elements of one figure corresponds a property of the corresponding set of elements of the corresponding, or transformed, figure.

When the correspondence is such that to an element of either figure corresponds one element and one only of the other, the correspondence is said to be *one-one*.

The study of the relations between such figures, when the correspondence is of a special type to be explained in Art. 3, constitutes what is called Projective Geometry.

2. Notation. Points will be denoted by Roman capitals $A, B, C, \ldots$

Straight lines will be denoted by small Roman letters $a, b, c, \ldots$; planes by small Greek letters $\alpha, \beta, \gamma, \ldots$.

Two elements are said to be *incident* if one lies in or passes through the other.

Thus if A lies on a, then a, A are incident.

If α contains a, then α, a are incident.

When two symbols are combined in the form of a product, the joint symbol denotes that element which is incident with the original two.

Thus AB denotes the straight line passing through the points A, B; Aa denotes the plane determined by the point A and the line a; $\alpha\beta$ denotes the line of intersection of the planes α, β; $\alpha\beta\gamma$ is the point of intersection of the three planes α, β, γ.

Such a joint symbol is not always interpretable. Thus ab has no meaning if a, b are lines in space which do not intersect. It has a meaning only if a, b are lines in one plane.

In dealing with corresponding figures, corresponding elements will invariably be *lettered* alike, the figure to which they belong being indicated by suffixes or by accents. Thus A_1 corresponds to A_2, a_1 to a_2 and so on. The student should be very careful to adhere rigidly to this practice, as random lettering obscures the correspondence of elements, which is their significant property and should be brought into prominence by every possible means.

The student is supposed familiar with the notion of a segment on a straight line as having *sense*, as well as magnitude. In this connection it should be noted that the sense of a segment will be indicated by the order of naming the letters.

Thus
$$AB = -BA,$$
and, whatever be the order of the points A, B, C on the line
$$AB + BC = AC.$$

When it is desired to consider merely the length of a segment AB, this will be written length AB or more shortly $|AB|$.

When the symbol AB is used it will in general be evident from the context whether the infinite straight line AB is meant, or only the segment AB.

3. Projection. Figures in space perspective. A fundamental method of obtaining corresponding plane figures is the following:

Let α_1 (Fig. 1) be any plane, V any fixed point outside it, α_2 any other plane. Let P_1 be any point of α_1; join VP_1, meeting α_2 at P_2. Then this construction establishes a correspondence between the points of the two planes, two corresponding points being always in a line through V. This correspondence is one-one; for, P_1 being known, P_2 is uniquely determined, and conversely (with certain cases of apparent exception to which we shall return presently).

Such a process of establishing a correspondence between the points of two planes is termed a projection. V is the *vertex of projection*, and we are said to project the points of α_1 from V upon α_2 or the points of α_2 from V upon α_1 according as the figure in the first or the second plane, respectively, is regarded as given; the plane *upon* which we project is spoken of as the *plane of projection*. The two figures thus connected are said to be in *space perspective*: they would appear coincident to an eye placed at V.

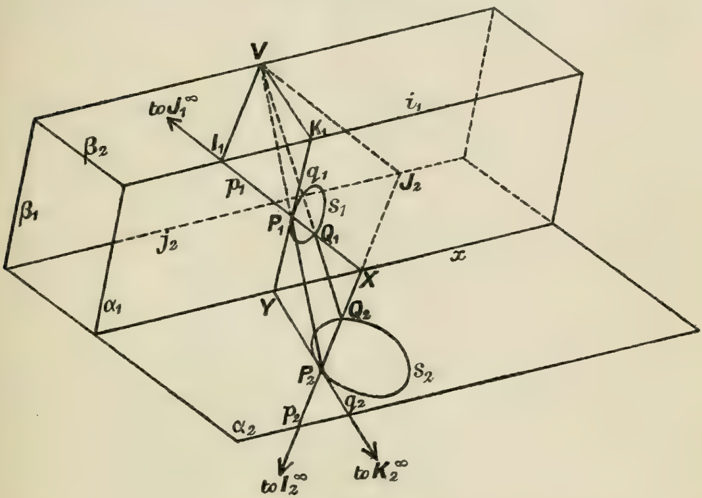


Fig. 1.

Let p_1 be a straight line of α_1; P_1 a point on p_1, P_2 its projection on α_2. VP_1 lies in the plane Vp_1, $\therefore$ P_2 lies in Vp_1, $\therefore$ P_2 lies on (Vp_1, α_2), which is a straight line p_2. Hence the locus of points corresponding to the points of p_1 is p_2. A straight line therefore corresponds to a straight line in projection.

We have immediately the results:

The join of two points has for its corresponding line the join of the two corresponding points.

The *meet* or intersection of two lines has for its corresponding point the meet of the two corresponding lines.

We may express this by saying that projection preserves unaltered properties of incidence, that is, if two elements are

incident in one figure, their corresponding elements are incident in the projected figure.

Let s_1 be any curve in the plane α_1, P_1, Q_1 two points on s_1: the corresponding points P_2, Q_2 lie on the curve s_2 which is the projection of s_1 on α_2. And when Q_1 approaches P_1, VQ_1 approaches VP_1 and therefore Q_2 approaches P_2. Therefore when P_1Q_1 approaches the tangent to s_1 at P_1, its corresponding line P_2Q_2 approaches the tangent to s_2 at P_2.

Projection therefore preserves unaltered properties of tangency, that is, to a tangent to a curve at a given point of one figure corresponds a tangent to the corresponding curve at the corresponding point of the other figure.

4. Elements at infinity. It has been stated that when P_1 is known, its projection P_2 is uniquely determined, and conversely. But this will be the case only if the line VP_1P_2 meets α_1 and α_2 at a finite distance. In this case both P_1 and P_2 are well determined points. If VP_1P_2 be parallel to the plane α_2 then in the language of Euclidean Geometry it does not meet α_2 at all and therefore no point P_2 exists.

In like manner if VP_1P_2 be parallel to α_1 no point P_1 exists.

In order to avoid the complications which would continually result from the necessity of considering such cases of exception, we introduce, by a convention, a set of new ideal elements, points, lines and plane, which are called the *elements at infinity*. By means of these elements the cases of exception are removed, and theorems can be stated in a more general manner.

We shall say that a given direction in space determines one *point at infinity*, through which all straight lines parallel to this direction are supposed to pass.

This gives a construction for the line joining P to a given point at infinity, viz. draw the parallel through P to the direction defining that point at infinity.

The student should note carefully that on any line there is *one* point at infinity only, not *two*. For if there were two points at infinity, a parallel to the line would pass through both of them, and two straight lines would intersect in more than one point, which would violate a fundamental postulate.

He may convince himself of the identity of the two opposite infinities on a line by imagining a ray through a point O outside the line and meeting the line at P to rotate continuously about O. P travels continuously along the line until the rotating ray passes through the position of parallelism, when P suddenly

passes from one extremity of the line to the other, showing that these opposite infinities are not separated.

To call attention to the fact that a point A lies at infinity, the symbol ∞ will be used as an index, thus A^∞.

Now let a be parallel to a plane α. Any plane through a cuts α in a line b parallel to a. If A^∞ be the point at infinity on a, A^∞ also lies on b and therefore on α. Therefore A^∞ is the intersection of a and α.

With this convention we may say that there is always one point corresponding to a given point of α_1 (Fig. 1).

For draw through V a plane β_1 parallel to α_1 and β_2 parallel to α_2. Let $i_1 = \alpha_1\beta_2$, $j_2 = \alpha_2\beta_1$.

All points P_1 of α_1 not on β_2 project into points P_2 of α_2 at a finite distance.

A point I_1 on i_1 projects into a point I_2^∞ of α_2 in the direction parallel to VI_1.

Similarly a point J_2 on j_2 projects into a point J_1^∞ of α_1 in the direction parallel to VJ_2.

Conversely to find the corresponding point or correspondent of a point J_1^∞ of α_1 in a direction parallel to p_1 (Fig. 1), we join VJ_1^∞, i.e. draw through V a parallel to p_1 to meet α_2 at J_2. VJ_2 being parallel to p_1 is parallel to α_1 and therefore lies in β_1. Hence J_2 lies on $\alpha_2\beta_1$, i.e. on j_2.

Hence the points at infinity in the plane α_1 project into the straight line j_2 of α_2 and similarly the points at infinity of α_2 project into the straight line i_1 of α_1.

Since the points at infinity in one plane correspond by projection to a straight line in the other plane, it is necessary to regard them as lying on a straight line.

The locus of the points at infinity in any plane is called the *line at infinity* in that plane.

Extending the notation already used we shall say that to i_1 corresponds i_2^∞ and to j_1^∞ corresponds j_2.

Two parallel planes are looked upon as intersecting in their common line at infinity.

It follows that all planes parallel to a given plane α pass through the line at infinity of α and are thus a particular case of a set of planes through a line. A line at infinity therefore corresponds to a definite *orientation**, as a point at infinity corresponds to a definite *direction*.

* The orientation of a plane is the lie of the plane relative to fixed directions. All parallel planes have the same orientation.

Two distinct points at infinity determine a line at infinity: for let the points be given by two non-parallel directions a, b; all planes parallel to both a and b are parallel among themselves, and their common line at infinity is the join of the two given points at infinity. Through any given point C at a finite distance one plane can be drawn containing such a line at infinity, namely the plane through C parallel to both a and b.

The aggregate of all points and lines at infinity is met by any other line in only one point and by any plane in only one line. The aggregate of points and lines at infinity therefore possesses the fundamental properties of a plane: and hence we speak of it as the *plane at infinity*.

The plane determined by three distinct non-collinear points A^∞, B^∞, C^∞ is the plane at infinity. For if it were a plane at a finite distance, A^∞, B^∞, C^∞ would lie on the line at infinity of this plane, which contradicts the hypothesis.

5. Vanishing points and lines. The line i_1 which corresponds to i_2^∞ is called the *vanishing line* of α_1. The line j_2 which corresponds to j_1^∞ is called the vanishing line of α_2.

The point I_1 of p_1, which corresponds to the point at infinity of p_2 is called the *vanishing point* of p_1; J_2 of p_2 which corresponds to the point at infinity of p_1 is called the vanishing point of p_2.

The vanishing point of a line is that point in which the line meets the vanishing line of its own figure.

Since VI_1 (Fig. 1) passes through I_2^∞ it is parallel to p_2. Hence:

The join of the vertex of projection to the vanishing point of any line is parallel to the projected line.

Considering two pairs of corresponding lines (p_1, p_2) (q_1, q_2), if K_1 be the vanishing point of q_1, VK_1 is parallel to q_2 and VI_1 is parallel to p_2. Hence the angle between p_2, q_2 = angle I_1VK_1, or:

The angle between two lines is equal to the angle subtended at the vertex of projection by the vanishing points of their corresponding lines.

6. Collineation. Any two corresponding lines p_1, p_2 lie in a plane π through V. If π meet the line of intersection x of α_1 and α_2 at a point X, X lies on π and α_1 and therefore on p_1; also X lies on π and α_2 and therefore on p_2, $\therefore$ p_1 p_2 meet at X on x. x is called the *axis of projection* or the *axis of collineation*.

Thus corresponding lines of figures in space perspective meet on the axis of collineation.

The points X of the axis are clearly self-corresponding, for: consider X as a point of α_1, VX meets α_2 at X, which therefore corresponds to itself. For this reason the points X are not distinguished by suffixes.

Conversely if two corresponding plane figures, such that the joins of corresponding points are corresponding lines, lie in different planes α_1, α_2, and possess the collineation property (namely that every pair of corresponding lines meet on a fixed line x), they are in space perspective.

To prove this we first of all observe that this line x can be no other than the intersection of the two planes; for there is no other locus where a line in α_1 can meet a line in α_2.

Let now P_1, Q_1 be any two points of one figure, P_2, Q_2 the corresponding points of the other figure.

Since P_1Q_1 meets P_2Q_2 $\therefore P_1$, Q_1, P_2, Q_2 are coplanar. Hence the lines P_1P_2, Q_1Q_2 are coplanar and must meet. Thus the joins of corresponding points meet in pairs. This is only possible if they all pass through one point. For if a third join R_1R_2 do not lie in the plane of P_1P_2, Q_1Q_2, it can meet both these lines only by passing through their intersection V. If it do lie in the plane of P_1P_2, Q_1Q_2, take a fourth join S_1S_2 which does not lie in this plane. S_1S_2 must pass through V. Hence if R_1R_2 is to meet S_1S_2, it also must pass through V. Hence all the joins of corresponding points pass through V and the figures are in space perspective.

7. Rabatting. For practical purposes, especially that of drawing projections, it is convenient to rotate one of the two planes about x until it coincides with the other plane. A figure in the former plane rotates with it, but is fixed in it. Such a process is termed *rabatting*.

8. Figures in plane perspective. If we have a figure 1 in the plane α_1 which is in space perspective with a figure 2 in the plane α_2 and we rabat the figure 2 upon the plane α_1 we obtain a new figure 3 in the plane α_1. The figure 3 is congruent with or superposable to the figure 2, but is in a different position and will be considered distinct from it.

We have now two corresponding figures, 1 and 3, in the *same plane.* It is important here to note carefully that the same point of the plane will in general have a quite different signifi-

cance (and be denoted by a different letter) according as we treat it as belonging to the figure 1 or to the figure 3.

The two figures 1 and 3 correspond to one another point by point and line by line. For the points and lines of 1 have a correspondence with the points and lines of 2 and these again with the points and lines of 3.

Also if p_1, p_2, p_3 be corresponding lines in the three figures, p_1 and p_2 meet at X on x. But the point X is not moved by a rotation about x. Hence p_3 passes through X. The two figures 1 and 3 are therefore coplanar corresponding figures possessing the collineation property.

Such figures are said to be *in plane perspective*, or *in homology*, or *homological*. x is then variously called the *axis of collineation*, or of *perspective*, or of *homology*.

9. Two figures in plane perspective are projections from two different vertices of a third figure in another plane.

Let there be two figures 1 and 2 in plane perspective in a plane α, so that their corresponding lines meet on an axis of collineation x.

Through x draw any other plane β. From a vertex V_1 project the figure 1 upon β. We obtain a figure 3. Also if p_1, p_2, p_3 are three corresponding lines of the three figures, p_1, p_2 meet at X on x and (because 1, 3 are in space perspective) p_1, p_3 also meet on x, and therefore must meet at X. Therefore p_2, p_3 meet at X on x. The figures 2, 3 are in different planes and possess the collineation property. Therefore by Art. 6 they are in space perspective from some vertex V_2. Figures 1 and 2 are therefore projections of figure 3 from V_1 and V_2.

It follows from the above that all the points of x are self-corresponding points.

Figures in plane perspective, like those in space perspective, possess vanishing lines, which correspond to the line at infinity of their plane treated as belonging to each of the two figures in turn.

Since the point at infinity on x is self-corresponding it lies on both vanishing lines. The latter are therefore parallel to the axis of collineation.

10. Pole of perspective. Using the notation of the last article, let A_1, A_2, A_3 (Fig. 2) be three corresponding points of the three figures. Then $A_1A_3V_1$ is a straight line and $A_2A_3V_2$ is a straight line. Since A_1V_1, A_2V_2 intersect at A_3 they are coplanar. Hence A_1A_2, V_1V_3 are coplanar and must intersect. But A_1A_2, lying in α, cannot meet V_1V_2 except at the point O where V_1V_2 meets α.

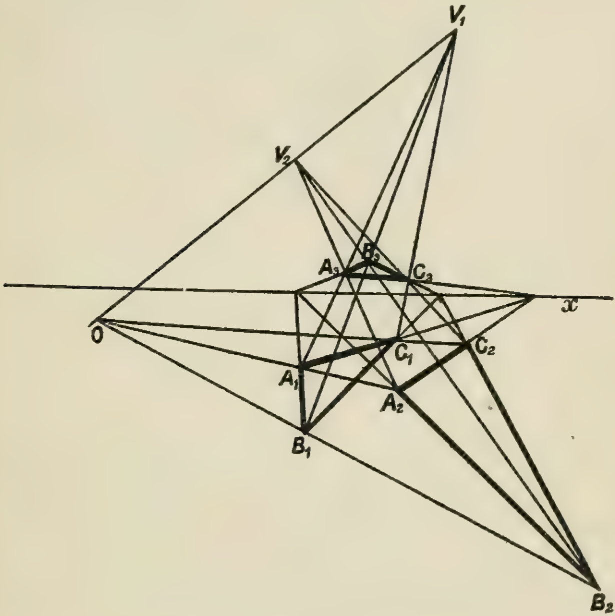


Fig. 2.

Hence the joins A_1A_2 of corresponding points of two figures in plane perspective pass through a fixed point O.

O is called the *pole* or *centre of collineation*, or of *perspective*, or of *homology*.

Conversely if two figures in the same plane α which correspond by points and lines possess the perspective property, i.e. are such that joins of corresponding points pass through a fixed pole O, they possess the collineation property and are in plane perspective.

For draw through O (Fig. 2) any straight line not in α and on

it take any two points V_1, V_2. Let A_1, A_2 be a pair of corresponding points. Then A_1A_2 meets V_1V_2, $\therefore A_1, A_2, V_1, V_2$ are coplanar and A_1V_1 meets A_2V_2 at some point A_3.

Now let b_1, c_1 be two lines through A_1; b_2, c_2 the two corresponding lines through A_2; and let a_1, a_2 be any other pair of corresponding lines. Write $a_1c_1 = B_1$; $a_1b_1 = C_1$; $a_2c_2 = B_2$; $a_2b_2 = C_2$; then as before $(B_1V_1, B_2V_2) = B_3$ and $(C_1V_1, C_2V_2) = C_3$. Let the plane $A_3B_3C_3$ cut α in x. The triangles $A_1B_1C_1$, $A_3B_3C_3$ are in space perspective from V_1; $A_2B_2C_2$ and $A_3B_3C_3$ are in space perspective from V_2. Therefore the triangles possess the collineation property; hence b_1b_2, c_1c_2, a_1a_2 all lie on x. But b_1, b_2, c_1, c_2 may be kept fixed, and the two points (b_1b_2), (c_1c_2) are sufficient to determine x. If then a_1, a_2 be any pair of corresponding lines whatever, (a_1a_2) lies on the fixed axis x.

11. Cylindrical projection. An important case of projection arises when the vertex V is itself at infinity. The projection is then said to be *cylindrical*; the lines joining corresponding points are all parallel and a curve and its projection are sections of a cylinder. The general projection, when V is at a finite distance, is sometimes called *central* or *conical* projection, a curve and its projection being here sections of a cone of vertex V.

In cylindrical projection, if i_2^∞ is the line at infinity of the plane α_2, i_1 is the intersection of α_1 by the plane $V^\infty i_2^\infty$. But the plane $V^\infty i_2^\infty$ is the plane at infinity. It will therefore meet α_1 in the line at infinity of α_1. Accordingly the vanishing lines are the lines at infinity in each plane, or in cylindrical projection lines at infinity correspond. From the projective standpoint this is the fundamental property of cylindrical projection.

If the joins of corresponding points in a cylindrical projection are perpendicular to the plane of projection, the projection is said to be *orthogonal*. Thus, if the plane of projection be horizontal, the points of this plane are vertically below their corresponding points. This type of projection is frequent in practical applications.

12. Locus of vertex of projection during rabatment. If ϕ_1, ϕ_2 be two figures in plane perspective in a plane α and having x for their axis of collineation; and if ϕ_2 be rotated about x through any angle θ into a position ϕ_3, the figures ϕ_1 and ϕ_3, having their corresponding lines meeting on x,

are in space perspective from some vertex V_1; and for a like reason ϕ_2 and ϕ_3 are in space perspective from some vertex V_2. As the angle θ is altered, V_1 and V_2 will alter.

In order to determine the vertex of projection, it is sufficient to know two pairs of corresponding points; the vertex is then the intersection of the two joins of corresponding points.

Let Fig. 3 represent the section of the plane α and of the plane of ϕ_3, which we may call β, by a plane perpendicular to x and passing through the pole of perspective O of ϕ_1 and ϕ_2. Let this plane meet x at X. Let the point at infinity on OX be denoted by J_1^∞ or I_2^∞ according as we treat it as belonging to ϕ_1 or to ϕ_2.

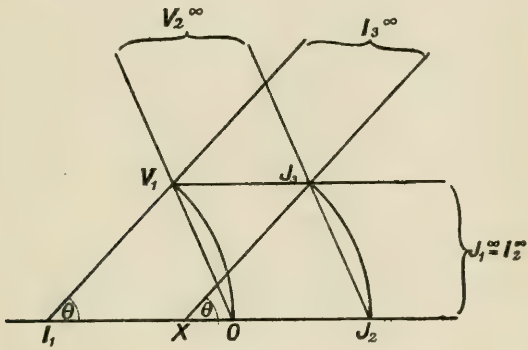


Fig. 3.

The points J_2 and I_1 lie on OX, since $J_2J_1^\infty$ and $I_1I_2^\infty$ each contain O.

The rotation θ brings J_2, I_2^∞ into positions J_3, I_3^∞ on a line through X making an angle θ with XO; and $XJ_3 = XJ_2$.

V_1 is now the intersection of $I_1I_3^\infty$ and $J_1^\infty J_3$; that is, it is the fourth angular point of the parallelogram of which XI_1, XJ_3 are sides. Hence $I_1V_1 = XJ_3 = XJ_2 =$ a constant length independent of the angle θ.

The locus of V_1 as θ varies is therefore a circle centre I_1, the ray I_1V_1 turning round at the same rate as the rotating figure.

V_2 is the intersection of J_2J_3 and $I_2^\infty I_3^\infty$.

$I_2^\infty I_3^\infty$ being the join of two points at infinity is the line at infinity in the plane of the paper. Therefore it meets J_2J_3 at infinity.

Thus V_2 is at infinity on J_2J_3.

Hence by Art. 10, ϕ_1, ϕ_2 being projections of ϕ_3 from V_1, V_2^∞ respectively, their pole of perspective O lies on $V_1V_2^\infty$, that is, on the parallel through V_1 to J_2J_3.

The triangles I_1V_1O, XJ_3J_2 having their sides parallel each to each, and $I_1V_1 = XJ_3$, are congruent. Therefore

$$I_1O = XJ_2 = I_1V_1.$$

Hence the locus of V_1 passes through O, or V_1 may be obtained by rotating O about the vanishing line of the fixed figure.

Conversely if the vertex of projection be known and one figure be rabatted upon the other, the pole of perspective may be constructed by rabatting the vertex—not about the axis of collineation, like the rest of the figure—but about the *vanishing line of the figure that does not move.*

That O should be on the locus of V_1 is almost intuitively evident if we consider the limiting position of V_1 when the rotation θ is made to approach zero. As a proof, however, this would be unsatisfactory, for a property which is true however small θ may be, is not necessarily still true when θ is actually zero.

13. Construction of figures in plane perspective. When the pole of perspective O and the axis of collineation x are given; and also a pair of corresponding points A_1 and A_2 (Fig. 4) the point P_2 corresponding to P_1 may be constructed as follows.

Join P_1A_1 meeting x at X. Where XA_2 meets OP_1 is P_2. For P_2 lies on OP_1 and A_2P_2, A_1P_1, being corresponding lines, meet on x.

In this way the second figure may be constructed from the first by points, or, by reversing the construction, the first figure may be derived from the second.

When O, x and a pair of corresponding lines a_1, a_2 (Fig. 4) are given, we construct the line p_2 corresponding to p_1 thus:

Join p_1a_1 to O. Where the join meets a_2 is the corresponding point p_2a_2. Join p_2a_2 to the point X, where p_1 meets x: then X is a point of p_2 $\therefore$ (X, p_2a_2) is p_2.

These constructions are simplified in practice if, instead of any two corresponding points or lines, one of the vanishing lines, say i_1, is given.

Take any point I_1 on i_1. By the property of the vanishing line and that of the pole of perspective, I_2 lies at infinity on

OI_1. If P_1I_1 meet x at X, X lies on $P_2I_2^\infty$, or P_2 lies on the parallel to OI_1 through X. Where this parallel meets OP_1 is P_2.

If any line p_1 be given, meeting i_1 at I_1, x at X, its correspondent p_2 is XI_2 and is therefore parallel to the join of the pole of perspective to the vanishing point of the given line. If p_1, q_1 be two given lines, the angle between p_2 and q_2 is equal to the angle subtended at O by the vanishing points of p_1, q_1 (cf. Art. 5).

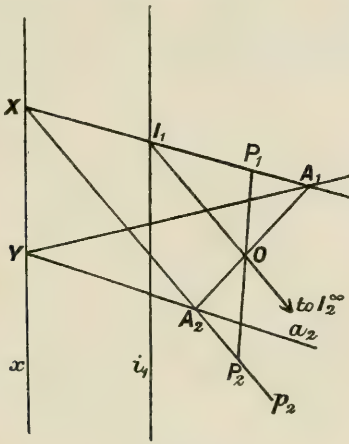


Fig. 4.

14. Particular cases of figures in plane perspective. If the axis of collineation x be at infinity, corresponding lines are parallel. The two figures in perspective are then similar and similarly situated and the centre of perspective becomes the centre of similitude.

Symmetry about an axis is likewise a particular case of plane perspective. The axis of symmetry is here the axis of collineation, for symmetrical lines meet on it. Also the join of two symmetrical points is perpendicular to the axis of symmetry. The pole of perspective is therefore at infinity in the direction perpendicular to the axis of symmetry.

A uniform *stretch* of a figure in any direction is a particular case of plane perspective. A stretch is defined as follows: if P_1 (Fig. 5) be any point and the line through P_1 parallel to a fixed direction called the direction of stretch meet a fixed line termed

the axis of stretch at X, the corresponding point P_2 is on XP_1 and is such that $XP_2 : XP_1 =$ a constant ratio, which is called the stretch-ratio. Clearly points on the axis of stretch are unchanged. Let p_1 (Fig. 5) meet this axis x at Z. If Q_1 be any other point on p_1, Q_1Y the distance of Q_1 from x measured in the direction of stretch, then $YQ_2 : YQ_1 = XP_2 : XP_1$ or, by a well known result in similar triangles, P_2, Q_2, Z are collinear, and the locus of Q_2 is a straight line p_2. Thus in a stretch a straight line corresponds to a straight line, and since p_1, p_2 meet on x, x is an axis of collineation. The pole of perspective is the point at infinity in the direction of stretch.

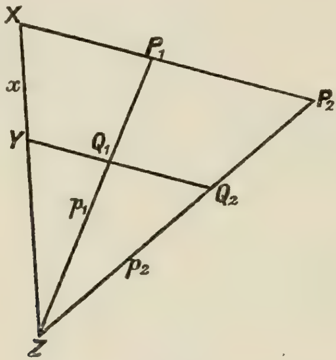


Fig. 5.

A translation without rotation of a figure in its own plane is a plane perspective transformation. For here also the joins of corresponding points are parallel to a fixed direction, namely that of the translation, and corresponding lines are parallel, that is, meet on the line at infinity. The latter is therefore the axis of collineation, and the pole of perspective is the point at infinity in the direction of the translation.

15. **Drawing of projections.** If it be required to draw on paper the projection upon a plane β of any given figure in a plane α from a given vertex V, or, what is the same thing, the section by β of a cone whose vertex is V and base the given figure, the method adopted in practice is to rabat the figure to be drawn upon the plane α about $\alpha\beta$. From the data of the

problem $\alpha\beta$ is known. Also drawing through V a plane parallel to β, this plane cuts α in the vanishing line i_1 of the given figure. The pole of perspective is then obtained by rabatting V about i_1 in the same sense that the projection is rabatted about $\alpha\beta$. We have now the pole of perspective, the axis of collineation and one vanishing line. The rabatted projection, which is of course in plane perspective with the original figure, may now be drawn by the rules given in the latter part of Art. 13. If at any stage the construction becomes awkward, so that lines or points employed in the construction come off the paper, two suitable corresponding points (or lines) may be found and the constructions given in the earlier part of Art. 13 can then be used.

If the projection be cylindrical, the construction by the vanishing line fails, for by Art. 11, both vanishing lines are then at infinity. Thus to a point I_1^∞ at infinity corresponds a point I_2^∞ also at infinity; and I_1^∞, I_2^∞ are in general distinct, since the axis of collineation is not here at infinity. Their join $I_1^\infty I_2^\infty$ is therefore the line at infinity; and O, which is on this line, is a point at infinity. Its position is then to be found by constructing, in any manner, some one pair of corresponding points A_1, A_2. O^∞ is then the point at infinity on A_1A_2. The construction for corresponding points which is given first in Art. 13 may then be used, remembering that, where a line is stated to be drawn "through O" in that construction, it should in the present case be drawn parallel to A_1A_2.

Notice that such a cylindrical projection, when rabatted into the plane of the original figure, is equivalent to a *stretch*.

16. Practical Example. A circle of radius 4 units and centre C lies in a horizontal plane α. V is a point 3 units vertically above a point A_1 of the circle. B_1 is a point of the circle 90° distant from A_1. The circle is projected from V on to a plane β passing through a line x in α which bisects CB_1 at right angles. The plane β is inclined at 60° to the horizontal plane. There are two such planes β. To completely define β we suppose that it is the one whose upper half is further from A_1.

Consider the plane γ which passes through V and is perpendicular to x. We shall need, for the practical construction, two figures (Fig. 6), one in γ which we shall call the elevation figure, and one in α which we shall call the plan figure. In the elevation figure the planes α, β appear as straight lines, viz. the lines in which they cut γ; these are called the traces of the planes on γ. Similarly in the plan figure γ appears as its trace on α. It is

convenient to place the figures one above the other, the two lines which represent $\alpha\gamma$ in the two figures being parallel, the points which represent the same points being on the same perpendiculars to $\alpha\gamma$.

Mark in the elevation figure the point A_1 and the point X where x meets $\alpha\gamma$. Through X draw a line making 60° with $\alpha\gamma$.

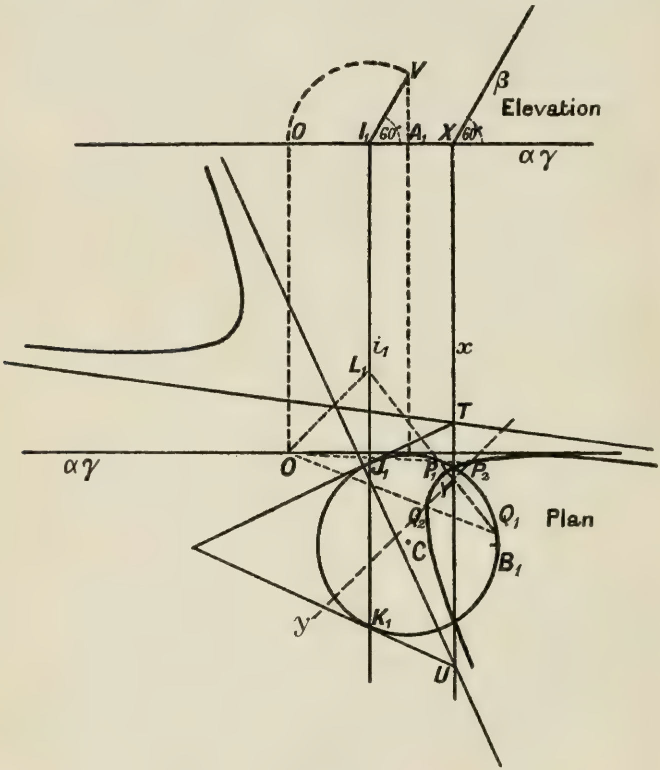


Fig. 6.

This is the trace of β. V is 3 units above A_1 in the elevation figure. Through V draw VI_1 parallel to the trace of β to meet $\alpha\gamma$ at I_1. I_1 is thus a point on the vanishing line of the original figure. Rotate V about I_1 counterclockwise into a position O on $\alpha\gamma$. O is the pole of perspective when the figure in plane β is

rabatted about x counterclockwise. Let the original figure and its rabatted projection be denoted by ϕ_1, ϕ_2 respectively. Then in the plan figure x is the axis of collineation, O the pole of perspective, the parallel i_1 to x through I_1 the vanishing line of ϕ_1.

To construct the figure corresponding to the circle we have the following method. Let L_1 be a fixed point on i_1. Take a variable point Y on x. Through Y draw a parallel y to OL_1. Join L_1Y meeting the circle at P_1, Q_1. OP_1, OQ_1 meet y at the points P_2, Q_2 corresponding to P_1, Q_1. By taking a number of parallels y we obtain a number of points on the projection of the circle. This projection is shown by the curve in Fig. 6.

The lines corresponding to the tangents at the points J_1, K_1 where i_1 meets the circle are important. These tangents at infinity or *asymptotes* (see later, Art. 34) are immediately constructed by drawing through the points T, U where the tangents to the circle at J_1, K_1 meet x, parallels to OJ_1, OK_1.

17. Repeated projection. The operation of projection may obviously be repeated any number of times, for the projection of a given figure may itself be projected upon a third plane and so on. The first and last figures will correspond point by point and line by line, and properties of incidence and tangency will be carried over from the first figure to the last, since they are preserved at each intermediate step.

Two figures which are derivable from one another by any number of such projections are said to be *projective*.

The projective property is a transitive one, that is, if a figure ϕ_1 be projective with ϕ_2 and ϕ_2 with ϕ_3, then ϕ_1 is projective with ϕ_3.

For a finite number of projections transform ϕ_1 into ϕ_2 and a finite number of projections transform ϕ_2 into ϕ_3. Applying the two sets of projections in succession a finite number of projections transform ϕ_1 into ϕ_3.

Two figures ϕ_1 and ϕ_2 which are in plane perspective are necessarily projective, for we have seen that a third figure ϕ_3 exists which is in space perspective with ϕ_1 and ϕ_2 from vertices V_1 and V_2 respectively (Art. 9). Thus by projecting ϕ_1 from V_1 into ϕ_3 and then ϕ_3 from V_2 into ϕ_2 we pass from ϕ_1 to ϕ_2 by two projective operations.

Conversely two coplanar projective figures ϕ_1 and ϕ_2 are not necessarily in plane perspective. This will appear from the examples in the following article.

18. Particular cases of projective figures.

A rotation of a plane figure about any axis perpendicular to its plane is a projective transformation.

Let OZ be the axis of rotation, θ the angle of rotation. Let OX_1 be any line through O lying in the plane α of the original figure, P_1 a point of that figure. Let β be the plane ZOX_1, which is perpendicular to α. Choose any vertex V_1 in space and project the figure 1 on to β, so that P_1 comes to P_3. Now rotate the figures 1 and 3, the vertex V_1 and the plane β as a rigid whole through an angle θ about OZ. V_1 takes up a position V_2, the plane β takes up a position γ, the figure 3 becomes a congruent figure 4 in the plane γ and the figure 1 becomes the figure 2. The figure 2 is therefore the projection from V_2 on to γ of the figure 4. Also the figures 3 and 4, being obtainable one from the other by rabatment about OZ, are in space perspective from a vertex V_3^∞ (Art. 12). Hence to derive 2 from 1 we project 1 from V_1 upon β as 3, then 3 from V_3^∞ upon γ as 4, then 4 from V_2 upon α as 2. The figures 1 and 2 are therefore projective.

They are not, however, in space perspective, for corresponding lines make a constant angle θ with each other. Thus the intersections of corresponding lines through two corresponding points P_1, P_2 is a circle at the circumference of which P_1P_2 subtend an angle θ. There is accordingly no axis of collineation.

Since any rigid displacement of a figure in its own plane may be broken up into a translation and a rotation, and since a translation is a projective transformation (Arts. 14, 17), any displacement of a figure in its plane is itself a projective transformation. And since the turning over of a plane figure is equivalent to constructing another figure symmetrical with the first, the axis of rotation being the axis of symmetry, and this last transformation is a plane perspective one (Art. 14), it follows that two coplanar figures, which can be superposed with or without turning over, are projective.

In like manner two similar figures, however placed, are projective. For by rotation, or by rotation combined with turning over, they can be similarly placed and they are then in plane perspective (Art. 14).

19. Problems in projection. It is often useful to be able to construct a projection so that the projected figure shall satisfy certain conditions. We will consider three of these.

I. To project a figure ϕ_1 so that a given line i_1 is projected to infinity. Thus i_1 is to be the vanishing line. Hence, the vertex V being arbitrarily selected, the plane of projection is any plane parallel to Vi_1.

II. To project a figure ϕ_1 so that a given line i_1 is projected to infinity and the angle between two given lines a_1, b_1 is projected into a given angle α.

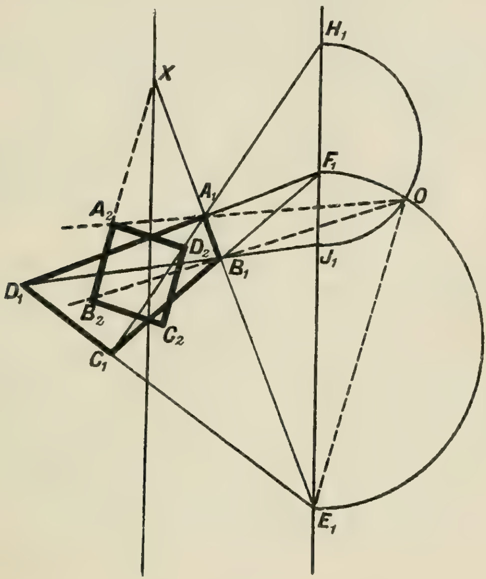


Fig. 7.

First solve the problem: to construct a plane perspective relation satisfying the required condition. Let A_1, B_1 be the points where a_1, b_1 respectively meet i_1. On A_1B_1 describe a segment of a circle containing an angle α. The pole of perspective O lies on this segment. Take for O any such point and for x any line parallel to i_1. This defines a plane perspective relation satisfying the given conditions. Now rotate O about i_1 through any angle θ into a position V, and at the same time rotate the plane of the original figure about x through the same angle θ into a position β. A projection from V on to β effects what is required.

III. To project a figure ϕ_1 so that a simple quadrilateral $A_1B_1C_1D_1$ (Fig. 7) becomes a square of given size. As in II we will solve the problem first for plane perspective.

Let E_1, F_1 be the intersections of opposite sides (A_1B_1, C_1D_1), (A_1D_1, B_1C_1) respectively; let G_1 (not marked in the figure) be the intersection of the diagonals (A_1C_1, B_1D_1).

Take E_1F_1 as vanishing line i_1; then E_2, F_2 are at infinity, and $A_2B_2C_2D_2$ is a parallelogram.

If the angle at G_2 (the angle between the new diagonals) is a right angle, the parallelogram $A_2B_2C_2D_2$ is a rhombus. If further any one of the angles at A_2, B_2, C_2, D_2 is a right angle, $A_2B_2C_2D_2$ is a square.

Describe on E_1F_1 a semicircle; if O lie on this semicircle the angles at A_1, B_1, C_1, D_1, which stand on E_1F_1, project into right angles.

Similarly if A_1C_1 meet E_1F_1 at H_1 and B_1D_1 meet E_1F_1 at J_1, O lies on a semicircle on J_1H_1. It is therefore the intersection of these two semicircles.

Now the side A_2B_2 must be parallel to OE_1, for E_1 is the vanishing point of A_1B_1. Also A_2 lies on OA_1, B_2 lies on OB_1. Place between OA_1, OB_1, parallel to OE_1, a length A_2B_2 equal to the side of the given square: this will be the line corresponding to A_1B_1. Where it meets A_1B_1 is a point X on the axis of collineation. Through X draw a parallel to the vanishing line E_1F_1; this is the axis x of collineation.

To obtain the required result by direct projection, rotate O about E_1F_1 through any angle into a position V, and project from V on to a plane through x parallel to VE_1F_1.

EXAMPLES I A.

1. Show that if two figures are in plane perspective the line joining any pair of corresponding points is a self-corresponding line.

2. Show that when two figures are in plane perspective there are two points in each figure, one of them being the pole of perspective, such that every angle at either point corresponds to an equal angle, and give a construction for the second point.

3. Show that when two figures are in space perspective there are two points in each figure such that every angle at any one of such points projects into an equal angle.

4. Prove that if two figures be in plane or in space perspective the images of the axis of collineation in the vanishing lines are corresponding lines such that corresponding segments on them are equal.

5. Prove (without using the property of the pole of perspective) that two figures in plane perspective are entirely given by the axis of collineation and two pairs of corresponding points. Deduce a construction for the point corresponding to a given point with the above data.

6. Given a pair of corresponding lines and the two vanishing lines of two figures in plane perspective, construct (a) the pole of perspective, (b) the point corresponding to any given point.

7. Given the pole of perspective, the axis of collineation and a pair of corresponding points of two figures in plane perspective, construct the two vanishing lines.

8. Construct the line corresponding to a given line, given the pole of perspective, the axis of collineation and a pair of corresponding points.

9. Prove that the distance of the pole of perspective from either vanishing line is equal to the distance of the axis of collineation from the other vanishing line.

10. Knowing one vanishing line, the axis of collineation and a pair of corresponding points, construct the pole of perspective.

11. Two figures, ϕ_1, ϕ_2 are in plane or in space perspective. Lines p_1, q_1 of ϕ_1 are parallel to fixed directions and are such that the angle between them corresponds to a constant angle in ϕ_2. Show that the intersection of $p_2 q_2$ describes a circle.

12. Show that, given any two triangles in a plane, a third triangle which is in plane perspective with each of them may be constructed in an infinite number of ways.

13. If a figure ϕ_1 is in plane perspective with ϕ_2 and ϕ_2 in plane perspective with ϕ_3, O_1, O_2 being the poles of perspective and x_1, x_2 the axes of collineation in the two cases, show that $O_1 O_2$ is self-corresponding in ϕ_1, ϕ_3 and find a point not on $O_1 O_2$ which is also self-corresponding in ϕ_1, ϕ_3.

14. Given any two triangles in space, a third triangle can always be found which is in space perspective with each of the original two.

15. Prove that two non-coplanar congruent figures are always projective.

16. Prove that two non-coplanar similar figures are always projective.

17. If in a plane perspective relation it is given that the pole of perspective and the axis of collineation are at infinity, show that the perspective relation must be equivalent to a translation without rotation in the plane.

18. Show how to project a given line to infinity and any two given angles into angles of given magnitude. Is this problem capable of solution in all cases?

19. Show how to project a given line to infinity and a given triangle into a triangle congruent with a given triangle.

20. A triangle ABC has its sides AB, AC cut at D and E by a parallel to the base. Show how to construct an equilateral triangle of given side which shall be in plane perspective with ABC, DE being taken as the vanishing line.

21. In Problem III of Art. 19 show that there are two possible positions of O and two possible positions of x and that these may be combined in pairs in four ways, so that there are four perspective relations giving a solution of the problem.

22. Show that if two figures are similar (but not necessarily similarly situated) the vanishing lines are at infinity.

23. Three coplanar triangles are two by two in perspective and have a common axis of collineation. Show that the poles of perspective are collinear.

24. Three coplanar triangles are two by two in perspective and have a common pole of perspective. Show that the axes of collineation are concurrent.

EXAMPLES I B.

1. Two figures in plane perspective have $x=0$ for axis of collineation. $A_1=(2, 0)$; $A_2=(2{\cdot}5, 2{\cdot}5)$; $B_1=(3, 0)$; $B_2=(1, 1)$ are pairs of corresponding points. Construct points corresponding to $P_1=(2, 3)$; $Q_2=(5, -4)$; I_1^∞ at infinity on $y=0$; J_2^∞ at infinity on $x+y=0$. Verify that A_1A_2, B_1B_2, P_1P_2, Q_1Q_2, I_1I_2, J_1J_2 all pass through a point.

2. The pole of perspective being the origin, the axis of collineation the line $x+2=0$ and the vanishing line of the figure ϕ_1 being $x=8$, construct the points of ϕ_2 corresponding to $(-\frac{1}{2}, 4)$, $(-1, -1)$, $(1, -2)$, $(2, 3)$; construct also the points of ϕ_1 corresponding to the same points.

3. Given the pole of perspective $(3, 0)$, the axis of collineation $x=0$ and the pair of corresponding lines $a_1\,(y=x)$ and $a_2\,(2y=x)$ construct by tangents the curve corresponding to the circle $x^2+y^2=4$ of the figure ϕ_1.

4. Two planes a_1, a_2 cut one another at an angle of 60°. On the plane bisecting the angle of 120° between them a vertex V is taken distant 4 inches from their line of intersection.

If a figure in a_1 is projected from V on to a_2 construct the vanishing lines of the figure in a_1 and the rabatted projection. If the axis of collineation be taken for axis of y and the foot of the perpendicular from V upon it as origin and if the positive half of the axis of x be the one nearer to V, find the points in a_2 corresponding to (2, 0), (5, 0), (3, 4) in a_1.

5. A pyramid 80 feet high stands on a square base of side 100 feet, the sides of the base running N. and S., E. and W. Draw the section of this pyramid by a plane at 30° to the horizontal passing through a line running from W.N.W. to E.S.E. through the S.W. corner of the pyramid, the plane rising as one moves N.

6. A right circular cone of semi-vertical angle 60° is cut by a plane making an angle of 30° with its axis and cutting that axis at a distance of 3 inches from the vertex. Draw the curve of section.

7. A horizontal square $ABCD$ of 2″ side is projected from a vertex 1·7″ above the corner A. Draw its projections upon the two planes through the diagonal BD inclined at 45° to the plane of the square.

8. A convex quadrilateral $ABCD$ is such that $AB=4''$, $AD=5''$, $CD=2''$, $CB=3''$, $AC=5''$. Find the pole and axis of collineation which will transform $ABCD$ into a square of side 1″ and draw this square.

9. The axis of x being taken as vanishing line, construct an equilateral triangle of side 2 units which is in plane perspective with the triangle whose vertices are (1, 2), (2·5, 2·5), (3, 1); and construct the pole of perspective and axis of collineation for this case.

10. A circular cylinder of radius 2″ is cut by a plane making an angle of 37° with its axis. Draw the section.

11. A horizontal circle is projected on to a vertical plane through its centre from a point at infinity on a ray inclined at 45° to the vertical and such that the vertical plane through it is inclined at 60° to the plane of projection.

12. The entrance of a skew tunnel is in the shape of a circular arch; the horizontal projection of the axis of the tunnel makes an angle of 15° with the normal to the plane of the arch and the axis itself slopes upwards at 30°. Draw the section of this tunnel by a horizontal plane.

CHAPTER II.

CROSS-RATIO; PROJECTIVE RANGES AND PENCILS.

20. Cross-ratio. Let A_1, B_1, C_1, D_1, etc. (Fig. 8) be a set of points on a straight line u_1. Let them be projected from

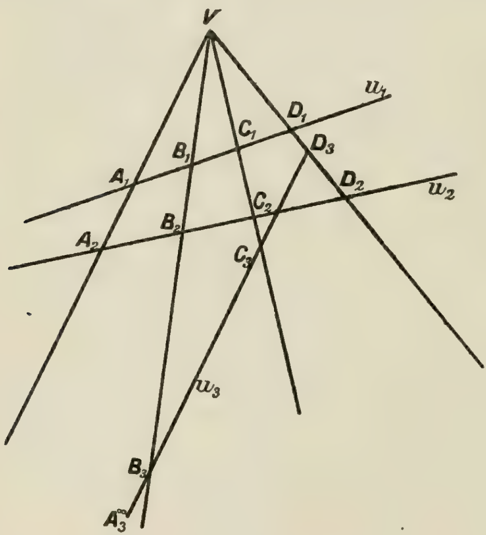


Fig. 8.

any vertex V into points A_2, B_2, C_2, D_2, ..., upon another straight line u_2.

We require to find a relation between the mutual distances of the points A_1, B_1, C_1, D_1, ... which will not be altered by projection.

Consider first the ratio of two segments.

$$\begin{aligned} A_1B_1 : A_1D_1 &= \triangle A_1VB_1 : \triangle A_1VD_1 \\ &= VA_1 . VB_1 \sin A_1VB_1 : VA_1 . VD_1 \sin A_1VD_1 \\ &= VB_1 \sin A_1VB_1 : VD_1 \sin A_1VD_1 . \end{aligned}$$

Note carefully that the above equation holds whatever the relative positions of A_1, B_1, D_1 (the signs of the segments being taken into account as explained in Art. 2), provided we introduce a corresponding convention as to the sign of the angles A_1VB_1, A_1VD_1.

In like manner

$$A_2B_2 : A_2D_2 = VB_2 \sin A_1VB_1 : VD_2 \sin A_2VD_2 .$$

$$\therefore (A_1B_1 : A_1D_1) \div (A_2B_2 : A_2D_2) = (VB_1 : VD_1) \div (VB_2 : VD_2).$$

The right hand side of the above equation is independent of the points A_1, A_2. It depends only on the bounding rays VB_1B_2, VD_1D_2. We may therefore replace A_1 by C_1, A_2 by C_2 without altering the value of the left hand side. We have then

$$(A_1B_1 : A_1D_1) \div (A_2B_2 : A_2D_2) = (C_1B_1 : C_1D_1) \div (C_2B_2 : C_2D_2)$$

or
$$\frac{A_1B_1 . C_1D_1}{A_1D_1 . C_1B_1} = \frac{A_2B_2 . C_2D_2}{A_2D_2 . C_2B_2}.$$

The expression $\dfrac{A_1B_1 . C_1D_1}{A_1D_1 . C_1B_1}$ is termed the *cross-ratio* or the *anharmonic ratio* of the four points A_1, B_1, C_1, D_1 taken in the given order and is denoted by the symbol $\{A_1B_1C_1D_1\}$. To remember it, note that the numerator is obtained by writing down the four points in the given order and the denominator is obtained from the numerator by interchanging the second and fourth elements. We have, then, from the last written equation the theorem:

The cross-ratio of any four collinear points is unaltered by projection.

21. Different cross-ratios of four points. If we take the points A_1, B_1, C_1, D_1 in a different order, we obtain in general a different cross-ratio. Since four letters may be written down in 24 different orders, we should expect 24 different cross-ratios. It will now be shown that only six of these are distinct.

First we shall prove that the cross-ratio of four points is unaltered if any two points be interchanged, provided the remaining two be also interchanged. Since under these circum-

stances the first point A_1 must necessarily be interchanged with some other, the three cases to be considered are therefore those where A_1 is interchanged with B_1, C_1 and D_1 respectively. We have to prove that

$$\{A_1B_1C_1D_1\} = \{B_1A_1D_1C_1\} = \{C_1D_1A_1B_1\} = \{D_1C_1B_1A_1\},$$

or, writing out the cross-ratios,

$$\frac{A_1B_1 \,.\, C_1D_1}{A_1D_1 \,.\, C_1B_1} = \frac{B_1A_1 \,.\, D_1C_1}{B_1C_1 \,.\, D_1A_1} = \frac{C_1D_1 \,.\, A_1B_1}{C_1B_1 \,.\, A_1D_1} = \frac{D_1C_1 \,.\, B_1A_1}{D_1A_1 \,.\, B_1C_1},$$

equalities which are obviously true.

It follows that distinct cross-ratios can be derived only from those permutations in which A_1 stands first. For, if we have any permutation in which A_1 does not stand first, it may be converted into a permutation in which A_1 does stand first by permuting A_1 with the leading element and interchanging the remaining two elements, and this without altering the cross-ratio.

We have then only six distinct cross-ratios, namely those in which A_1 stands first, the remaining three B_1, C_1, D_1 being permuted in all possible ways.

To find the relation among these ratios, project A_1 to infinity, that is, cut the four rays through V by a straight line u_3 (Fig. 8) parallel to VA_1. We have by Art. 20

$$\{A_1B_1C_1D_1\} = \{A_3{}^\infty B_3C_3D_3\} = \frac{A_3{}^\infty B_3 \,.\, C_3D_3}{A_3{}^\infty D_3 \,.\, C_3B_3}.$$

But
$$\frac{A_3{}^\infty B_3}{A_3{}^\infty D_3} = \frac{A_3{}^\infty D_3 + D_3B_3}{A_3{}^\infty D_3} = 1 + \frac{D_3B_3}{A_3{}^\infty D_3} = 1,$$

since the ratio of a finite to an infinite segment is zero.

$$\therefore \ \{A_1B_1C_1D_1\} = \frac{C_3D_3}{C_3B_3} = \lambda, \text{ say} \quad \ldots\ldots\ldots\ldots(1).$$

Interchange even letters D and B. Then

$$\{A_1D_1C_1B_1\} = \frac{C_3B_3}{C_3D_3} = \frac{1}{\lambda} \quad \ldots\ldots\ldots\ldots\ldots\ldots(2).$$

Interchange middle letters B and C,

$$\{A_1C_1B_1D_1\} = \frac{B_3D_3}{B_3C_3} = \frac{D_3B_3}{C_3B_3} = \frac{D_3C_3 + C_3B_3}{C_3B_3}$$

$$= 1 - \frac{C_3D_3}{C_3B_3} = 1 - \lambda \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(3).$$

Interchange in (2) the middle letters,

$$\{A_1C_1D_1B_1\} = 1 - \frac{1}{\lambda} = \frac{\lambda - 1}{\lambda} \quad \ldots\ldots\ldots\ldots\ldots(4).$$

Interchange second and fourth letters in (3),

$$\{A_1D_1B_1C_1\} = \frac{1}{1-\lambda} \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots(5).$$

Interchange second and fourth letters in (4),

$$\{A_1B_1D_1C_1\} = \frac{\lambda}{\lambda - 1} \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots(6).$$

These give the six distinct cross-ratios of four points.

22. Cross-ratio of four rays. From Art. 20 it follows that all transversals, that is, all straight lines which cut a set of four rays or lines through a point, have a constant cross-ratio. This cross-ratio is therefore a property of the set of four rays and is called the cross-ratio of the four rays.

The analytical expression for the cross-ratio of four such rays is easily written down. For

$$\begin{aligned}\frac{A_1B_1 \,.\, C_1D_1}{A_1D_1 \,.\, C_1B_1} &= \frac{\triangle A_1VB_1}{\triangle A_1VD_1} \cdot \frac{\triangle C_1VD_1}{\triangle C_1VB_1} \\ &= \frac{VA_1 \,.\, VB_1 \sin A_1VB_1 \,.\, VC_1 \,.\, VD_1 \sin C_1VD_1}{VA_1 \,.\, VD_1 \sin A_1VD_1 \,.\, VC_1 \,.\, VB_1 \sin C_1VB_1} \\ &= \frac{\sin A_1VB_1 \,.\, \sin C_1VD_1}{\sin A_1VD_1 \,.\, \sin C_1VB_1},\end{aligned}$$

the signs of the angles being attended to; A_1VB_1 being measured by the rotation (positive counterclockwise) which brings VA_1 to VB_1.

Since four concurrent rays project into four concurrent rays and transversals into transversals, it follows from the permanence of cross-ratio of four points in projection that the cross-ratio of four rays is likewise unaltered by projection.

The cross-ratio of four rays *abcd* will be denoted by $\{abcd\}$. If the rays be OA, OB, OC, OD, it will also be denoted by $O\{ABCD\}$.

23. Ranges and pencils. A *range* is a set of points on a straight line. A *flat pencil*, or shortly a *pencil*, is a set of rays through a point which is the *vertex* or *centre* of the pencil.

Ranges and pencils are called one-dimensional elementary geometric forms. The straight line containing the range or the

vertex of the pencil is spoken of as the *base* of the form: the component points or rays are spoken of as its *elements*.

A form may be denoted by bracketing a number of its elements, thus $(A_1B_1C_1D_1)$ denotes a range of which A_1, B_1, C_1, D_1 are points. More frequently it will be denoted by taking a typical element and enclosing it in square brackets. Thus $[P]$ is a range of which the point P, which is then considered variable, is the typical element: $[p]$ is a pencil of which p is the typical ray, or $O[P]$ is a pencil with vertex O, of which OP is the typical ray. Often when no confusion is likely to result, a form will be denoted simply by its base, thus u will denote a range on the line u, U a pencil whose vertex is U.

24. Projective ranges and pencils. The elements of two forms may be made to correspond, each to each. When the forms are of the same type, that is, when both are ranges or both pencils, they are said to be projective when the correspondence can be established by means of a finite number of projective operations. It will then follow from Arts. 20, 22 that projective ranges and pencils are also equi-anharmonic, that is, any four elements of one form have the same cross-ratio as the four corresponding elements of any other form projective with the first.

An important particular case of projective ranges and pencils is when the two ranges are sections of the same pencil by two different transversals, or when the two pencils are obtained by joining up the points of the same range to two different vertices. In the first case the joins of corresponding points of the two ranges pass through a fixed point: in the second case the meets of corresponding rays of the two pencils lie on a fixed line. Two such ranges and pencils are said to be *perspective*: they are clearly particular cases of figures in plane or in space perspective, and are therefore projective.

If two ranges be perspective the point where their bases intersect is self-corresponding, and if two coplanar pencils be perspective the ray joining the two vertices is self-corresponding.

Similar ranges are corresponding ranges in which corresponding segments are proportional.

Equal ranges or pencils are ranges and pencils which can be superposed so that corresponding elements coincide.

Equal pencils in one plane are said to be *directly*, or *oppositely*, equal according as they can, or cannot, be superposed without being turned over.

Since it has been shown (Art. 18) that congruent and similar figures are particular cases of projective figures, it follows that

similar ranges are projective and also that equal ranges and equal pencils are projective.

In two similar ranges the points at infinity correspond. For since $A_1B_1 : A_2B_2 =$ a finite ratio λ, if A_1B_1 is infinite, so must A_2B_2 be infinite. Hence if A_1, A_2 be points at a finite distance and B_1 a point at infinity, B_2 is at infinity.

Conversely projective ranges in which the points at infinity correspond are similar. Let $A_1B_1C_1I_1^\infty$, $A_2B_2C_2I_2^\infty$ be corresponding groups of four points of two such ranges, then, since the cross-ratio is unaltered,

$$\frac{A_1B_1 \,.\, C_1I_1^\infty}{A_1I_1^\infty \,.\, C_1B_1} = \frac{A_2B_2 \,.\, C_2I_2^\infty}{A_2I_2^\infty \,.\, C_2B_2},$$

and remembering (Art. 21) that $\dfrac{C_1I_1^\infty}{A_1I_1^\infty} = 1$ and $\dfrac{C_2I_2^\infty}{A_2I_2^\infty} = 1$, we have

$$A_1B_1 : C_1B_1 = A_2B_2 : C_2B_2,$$

or the ranges are similar.

Projective ranges and pencils may be *cobasal*, that is, a projective correspondence can be established between points of the same line, or between rays passing through the same vertex. In this case a particular point of the base has a different significance, according as we consider it to belong to one range, or to the other: similarly a particular ray through the vertex has a different significance, according as it belongs to one or to the other of the two pencils. Such ranges are termed *collinear* and such pencils *concentric*.

Sections of two projective pencils $[p_1]$, $[p_2]$ by transversals v, w are projective.

For in the set of projections which transform $[p_1]$ to $[p_2]$ let a line u_1 not belonging to $[p_1]$ transform into u_2.

The range $u_1[p_1]$ is projective with $u_2[p_2]$.

But $v[p_1]$ is perspective and $\therefore$ projective with $u_1[p_1]$, and $w[p_2]$ is perspective and $\therefore$ projective with $u_2[p_2]$.

Hence $v[p_1]$ is projective with $w[p_2]$.

Similarly if $[P_1]$, $[P_2]$ be two projective ranges, O, S any two vertices, the pencils $O[P_1]$, $S[P_2]$ are projective.

For in the set of projections which transform $[P_1]$ to $[P_2]$ let a point U_1 not belonging to $[P_1]$ transform into U_2.

The pencil $U_1[P_1]$ is projective with the pencil $U_2[P_2]$.

But $O[P_1]$ is perspective and $\therefore$ projective with $U_1[P_1]$, and $S[P_2]$ is perspective and $\therefore$ projective with $U_2[P_2]$.

Hence $O[P_2]$ is projective with $S[P_2]$.

25. Two cobasal projective forms are identical if they have three elements self-corresponding.

Consider two ranges. Let A, B, C be the self-corresponding points, P_1, P_2 any two corresponding points.

Then
$$\{ABCP_1\} = \{ABCP_2\},$$
$$\therefore \frac{AB \,.\, CP_1}{AP_1 \,.\, CB} = \frac{AB \,.\, CP_2}{AP_2 \,.\, CB},$$
$$\frac{CP_1}{AP_1} = \frac{CP_2}{AP_2},$$
$$\therefore \frac{CA + AP_1}{AP_1} = \frac{CA + AP_2}{AP_2},$$
$$\therefore CA \,.\, AP_2 = CA \,.\, AP_1.$$

CA is not zero, since by hypothesis the points A, B, C are distinct, $\therefore AP_1 = AP_2$ or P_1, P_2 are coincident. Hence every point is self-corresponding and the ranges are identical.

Consider now two concentric pencils. They determine on any line two collinear projective ranges. If three rays of the pencils are self-corresponding, three points of the ranges are self-corresponding. Therefore every point of the ranges is self-corresponding and in consequence every ray of the pencils is self-corresponding.

It follows that two distinct cobasal projective forms cannot have more than two self-corresponding elements.

26. Construction of projective ranges and pencils from corresponding triads.

I. **Ranges.** If the given ranges are in different planes, or if they are in the same straight line, let them first of all be projected into two coplanar non-collinear ranges. We need therefore consider only the problem of establishing a projective correspondence between two ranges of the latter type.

Let u_1, u_2 (Fig. 9) be the two ranges. Let A_1, B_1, C_1 be any three given points of u_1, A_2, B_2, C_2 the three corresponding points of u_2. Join A_1A_2 and on it take any two points S_1, S_2. Join S_1B_1, S_2B_2 meeting at B_3, and S_1C_1, S_2C_2 meeting at C_3. Let $B_3C_3 = u_3$ meet A_1A_2 at A_3. Then A_1, B_1, C_1 are perspective with A_3, B_3, C_3 from S_1 and A_3, B_3, C_3 in turn are perspective with A_2, B_2, C_2 from S_2.

Take now P_1 any point of the range u_1. Project P_1 from S_1 as vertex into P_3 on u_3 and then P_3 from vertex S_2 into P_2 on u_2. The ranges $[P_1]$, $[P_2]$ are projective. The range $[P_2]$ is therefore

projective with the original range u_2 and lies in the same straight line with it. But these two collinear ranges have clearly A_2, B_2, C_2 for three self-corresponding points. Hence the range $[P_2]$ is identical with the original range u_2 and the construction given connects corresponding points of the two ranges u_1, u_2.

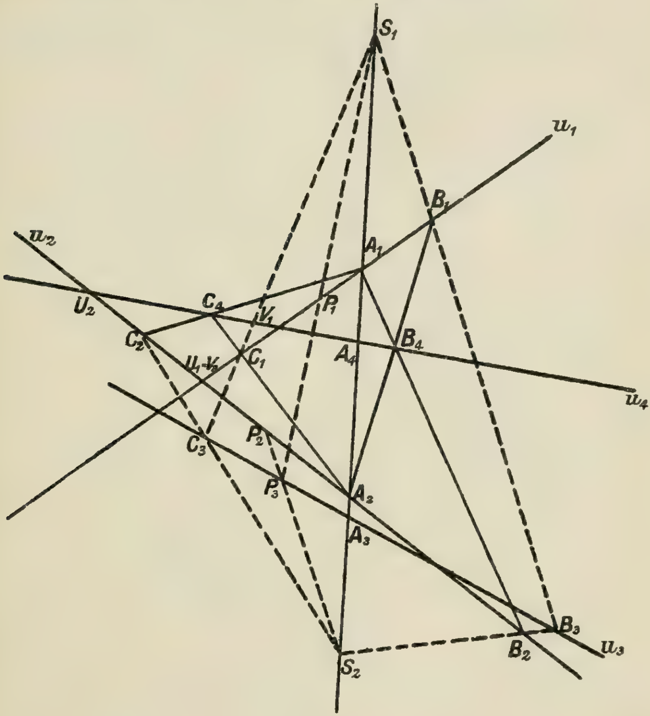


Fig. 9.

II. **Pencils.** First of all, if the two pencils are not already coplanar and non-concentric, project them into pencils which are coplanar and non-concentric. We shall then consider the two pencils to be of this type.

Let U_1, U_2 (Fig. 10) be two such pencils. Let a_1, b_1, c_1 be any three given rays of U_1 corresponding to a_2, b_2, c_2 of U_2. Through a_1a_2 $(=A)$ draw any two transversals s_1, s_2 meeting b_1, c_1 at

B_1, C_1 and b_2, c_2 at B_2, C_2 respectively. Let B_1B_2, C_1C_2 meet at U_3. Let U_3A, U_3B_1, U_3C_1 be a_3, b_3, c_3. The sets of rays $a_1b_1c_1$, $a_3b_3c_3$ are perspective and so are $a_3b_3c_3$, $a_2b_2c_2$. If p_1 be any ray of u_1 (not shown in the figure) meeting s_1 at P_1, join $U_3P_1 = p_3$ meeting s_2 at P_2 and join $U_2P_2 = p_2$. The pencils $[p_1]$, $[p_2]$ are projective. Hence $[p_2]$ and the original pencil U_2 are projective and have three self-corresponding rays a_2, b_2, c_2. They are therefore identical, and the given construction connects all corresponding rays of the two original pencils.

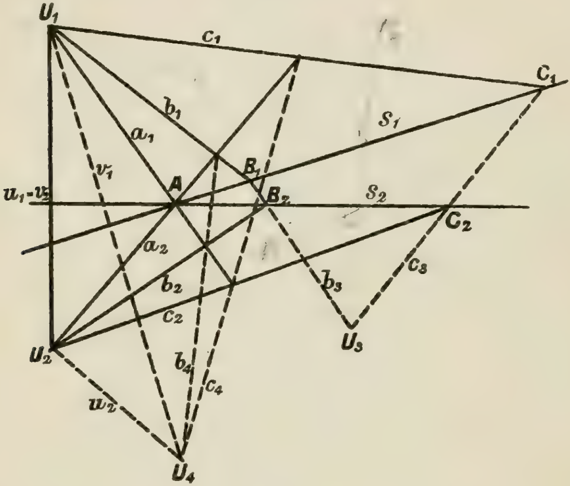


Fig. 10.

It follows from the above constructions:

(*a*) That the relation between two projective forms is entirely determined as soon as three corresponding pairs of elements are given.

(*b*) That a projective relation between two like forms can always be established in which three arbitrary elements of one shall correspond to three arbitrary elements of the other, which is sometimes expressed by saying that groups of three elements are always projective.

(*c*) That a projective relation between two like forms can always be established in which any four elements of the one

correspond to four elements of the other having the same cross-ratio.

For let A_1, B_1, C_1, D_1; A_2, B_2, C_2, D_2 be two sets of four points of ranges u_1, u_2. Let the projective relation which transforms A_1, B_1, C_1 into A_2, B_2, C_2 transform D_1 into D_2'. Then

$$\{A_1B_1C_1D_1\} = \{A_2B_2C_2D_2\} = \{A_2B_2C_2D_2'\}$$

and from the last equality it follows as in Art. 25 that $D_2 = D_2'$. A corresponding proof holds for pencils.

The above constructions fail if in Fig. 9 $A_1 = A_2$, or in Fig. 10 $a_1 = a_2$, that is, if the two given forms have one element self-corresponding. The line A_1A_2 and the point a_1a_2 are then indeterminate. In such a case the point A $(= A_1 = A_2)$ must be the meet of the bases and the ray a $(= a_1 = a_2)$ must be the join of the vertices. A simpler construction can then be given. For let B_1B_2 meet C_1C_2 at V, then AB_1C_1 and AB_2C_2 are perspective from V. Similarly if the join of b_1b_2, c_1c_2 be v, ab_1c_1, ab_2c_2 are perspective, corresponding rays meeting on v. The two given ranges or pencils are then perspective and we have the important result:

If two projective ranges or flat pencils, which are not cobasal, have a self-corresponding element, they are perspective.

27. Harmonic forms. Since any three collinear points may be projected into any three other points, three points A, B, C on a line c' (Fig. 11) may be projected into the same points with two of them, say A and C, interchanged.

To effect this, draw any line a' through B and from any vertex S in the plane $a'c'$ project A, B, C upon a' as A', B, C'. Let $(A'C, AC') = T$. Then if we project A', B, C' from T upon c', they project into C, B, A. The double operation has therefore interchanged A and C.

The two triads ABC, CBA define two projective collinear ranges on c'. These two ranges have already a self-corresponding point B. They have therefore at most one other point D which corresponds to itself.

This point D is the point where ST meets c'. For if ST meet a' at D', D projects from S on a' into D' and D' projects back from T on c' into D.

Hence $$\{ABCD\} = \{CBAD\}$$

and D is the only point satisfying this condition.

When four points are such that they are projective with themselves, two of them being interchanged, they are said to be

harmonic, or to form a *harmonic range*, and the two which are interchanged are said to be *harmonically conjugate* with regard to the other two.

By Art. 21, interchanging both A and C, B and D

$$\{CBAD\} = \{ADCB\}.$$

Hence $$\{ADCB\} = \{ABCD\}.$$

It follows from (*c*) of Art. 26 that if A, B, C, D can be projected into C, B, A, D, they can be projected into A, D, C, B.

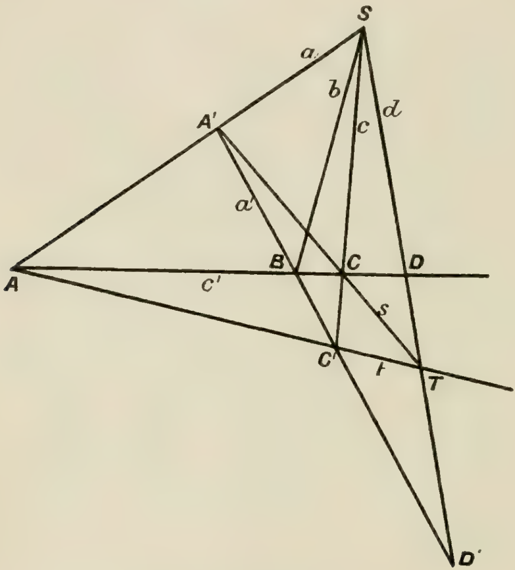


Fig. 11.

So that if A, C are conjugate with regard to B, D, so are B, D with regard to A, C.

If we join the points of a harmonic range to a point outside the range, we obtain a pencil of four rays possessing the same property, namely that it is projective with itself, two rays being interchanged. The interchangeable rays are termed conjugate and the pencil is termed a *harmonic pencil*.

28. Cross-ratio of four harmonic elements. Let

λ be the cross-ratio of four harmonic elements, say four points A, B, C, D of a range.

If $\{ABCD\} = \lambda$, then by Art. 21 $\{ADCB\} = \frac{1}{\lambda} = \{CBAD\}$.

Hence $$\lambda = \frac{1}{\lambda}, \quad \text{or } \lambda = \pm 1.$$

Now if λ were $+1$, we should have

$$\begin{aligned} AB \,.\, CD &= AD \,.\, CB \\ &= (AB + BC + CD)\, CB, \\ AB\,(BC + CD) &= (BC + CD)\, CB, \\ (AB + BC)(BC + CD) &= 0, \\ AC \,.\, BD &= 0. \end{aligned}$$

That is, either C and A, or B and D coincide. But this is not the case, by hypothesis. Hence the cross-ratio of four harmonic elements, in which conjugate elements are not coincident, is -1.

The cross-ratio of a harmonic pencil is also -1, since such a pencil stands on a harmonic range.

It follows at once that every transversal cuts a harmonic pencil in a harmonic range and also that four harmonic elements are necessarily projective with four other harmonic elements, since the two sets have the same cross-ratio.

The relation $\{ABCD\} = -1$ can be put into two other different forms, which are of great importance.

We have $$AB \,.\, CD + AD \,.\, CB = 0,$$
i.e. $$AB\,(CA + AD) + AD\,(CA + AB) = 0;$$
or $$AB \,.\, AC + AD \,.\, AC = 2 \,.\, AB \,.\, AD,$$
and dividing by $AB \,.\, AC \,.\, AD$
$$\frac{1}{AB} + \frac{1}{AD} = \frac{2}{AC}.$$

To get the other form, let O be the point midway between two conjugates, say A and C. Substituting into the relation
$$AB \,.\, CD + AD \,.\, CB = 0,$$
we have
$$(AO + OB)(CO + OD) + (AO + OD)(CO + OB) = 0,$$
$$2 \,.\, AO \,.\, CO + (OB + OD)(AO + CO) + 2 \,.\, OB \,.\, OD = 0.$$
But $$AO = OC = -CO, \quad \therefore\ AO + CO = 0,$$
$$\therefore\ OB \,.\, OD = -AO \,.\, CO = OA^2.$$

When $\{ABCD\} = -1$, $AB : AD = -CB : CD$, or the points A and C divide BD internally and externally in the same ratio. Hence by Euclid VI. 3 the two bisectors of the angles formed by a pair of straight lines are harmonically conjugate with regard to the two given lines.

Conversely if, in a harmonic pencil, one pair of conjugate lines are at right angles, they bisect the angles formed by the other pair. For let a, c be at right angles. Then if b, d be not equally inclined to a, c let b, d' be equally inclined to a, c: then $\{a, b, c, d'\} = -1 = \{a, b, c, d\}$, $\therefore\ d = d'$, that is b, d are equally inclined to a, c.

If one of the points of a harmonic range be at infinity its conjugate is midway between the other two. For, let A^∞ be this point, then

$$\frac{A^\infty B \,.\, CD}{A^\infty D \,.\, CB} = -1, \quad \text{or} \quad \frac{CD}{CB} = -1,$$

that is $BC = CD$ or C bisects BD.

If $B = D$, or $C = A$, the definition of Art. 27 apparently leads to an indeterminate result. Let us agree that the equation

$$AB \,.\, CD + AD \,.\, CB = 0$$

shall hold in all cases. If we now put $D = B$, we have

$$2AB \,.\, CB = 0.$$

Hence either $AB = 0$ or $CB = 0$, that is, either A or C coincides with B and D. That the same result holds for pencils is easily seen on cutting by a transversal.

29. Harmonic properties of the complete quadrilateral and quadrangle. A complete quadrilateral is the figure formed by four straight lines a, b, c, d, called its sides. It has six vertices ab, ac, ad, bc, bd, cd formed by taking meets of sides in pairs. The three pairs of vertices ab, cd; ac, bd; ad, bc such that the two in each pair do not lie on a common side are termed opposite vertices; the three lines joining them are called the diagonals of the quadrilateral. The triangle formed by them is the diagonal triangle of the quadrilateral.

A complete quadrangle is the figure formed by four points A, B, C, D called its vertices. It has six sides AB, AC, AD, BC, BD, CD formed by taking joins of vertices in pairs. The three pairs of sides AB, CD; AC, BD; AD, BC such that the two in each pair do not pass through a common vertex are termed opposite sides. Their three meets are called the diagonal points

of the quadrangle. The triangle formed by them is the diagonal triangle.

The harmonic properties of the complete quadrilateral and quadrangle are as follows:

I. The two vertices of a complete quadrilateral on any diagonal are harmonically conjugate with regard to the two vertices of the diagonal triangle on that diagonal.

II. The two sides of a complete quadrangle through a diagonal point are harmonically conjugate with regard to the two sides of the diagonal triangle through that diagonal point.

To prove these results, refer to Fig. 11. Here AA', $A'C$, CC', $C'A$ are the four sides of a complete quadrilateral, of which $A'C'$, AC, ST are the three diagonals. The diagonal AC is divided harmonically at B and D (Art. 27). But B and D are the points where AC is met by the other two diagonals. The result for the other diagonals follows by symmetry.

Again A, C', C, A' are the four vertices of a complete quadrangle, of which S, B, T are the three diagonal points. The two sides through S, SA and SC, are harmonically conjugate with regard to SB and SD (since A, C are harmonically conjugate with regard to B, D). But SB, SD are the two sides of the diagonal triangle through S.

From the above properties we obtain the following constructions for the element harmonically conjugate to a given element with regard to two given elements.

I. Through the point B, to which a conjugate is required with regard to A and C, draw any line and on it take any two points A', C' (Fig. 11). Join AA', CC' meeting at S, AC', $A'C$ meeting at T. TS meets the original line in the point D required.

II. On the ray $SB = b$, to which a conjugate is required with regard to $SA = a$, $SC = c$, take any point B, and through it draw any two lines a', c'. Let $s =$ join of aa', cc', $t =$ join of ac', $a'c$. The join of ts ($= T$) to the vertex S gives the ray d required.

In the above cases it is often said that D is a fourth harmonic to A, B, C and d a fourth harmonic to a, b, c, respectively.

30. Cross-axis and cross-centre of coplanar projective ranges and pencils. If in construction I of Art. 26 (Fig. 9) S_1 be taken at A_2 and S_2 at A_1 we obtain an intermediate range u_4 perspective with u_1 from A_2 and with u_2 from A_1.

To construct by means of u_4 the points which correspond to the point of intersection of u_1, u_2. Let this point considered

as a point of u_1 be called U_1, and considered as a point of u_2 be called V_2.

A_2U_1 meets u_4 at U_4; A_1U_4 meets u_2 at U_2. But A_2U_1 is itself u_2. Therefore $U_4 = U_2 = u_2u_4$. In like manner $V_1 = u_1u_4$. Now the projective relation between the ranges being given, U_2, V_1 are fixed points and therefore $u_4 = U_2V_1$ is a fixed line. $A_1B_1C_1$, $A_2B_2C_2$ may be any corresponding triads whatever of the given ranges.

It follows that if A_1A_2, B_1B_2 be any two pairs of corresponding points of two projective ranges the meet of cross-joins (A_1B_2, A_2B_1) lies on a fixed straight line. This straight line may be termed the *cross-axis* of the two projective ranges.

Similarly if in construction II of Art. 26 (Fig. 10) s_1 be taken coincident with a_2 and s_2 with a_1, the vertex of the intermediate pencil is a point U_4. If we now consider U_1U_2 and treat it as a ray u_1 of the pencil U_1, it meets a_2 at U_2, $\therefore$ $U_4U_2 = u_4$. But the pencils U_4, U_2 being perspective, U_4U_2 is self-corresponding, hence $U_4U_2 = u_2$. Similarly if $U_1U_2 = v_2$, $U_1U_4 = v_1$. Hence U_4 is the intersection of the two rays corresponding to U_1U_2; U_4 is therefore a fixed point. $a_1b_1c_1$, $a_2b_2c_2$ are any corresponding triads. Hence if a_1a_2, b_1b_2 be any two pairs of corresponding points of two projective pencils the join of cross-meets (a_1b_2, a_2b_1) passes through a fixed point. This fixed point may be termed the *cross-centre* of the two projective pencils.

If the ranges (or pencils) in the above theorems be perspective the reasoning employed fails, for then u_1u_2 (Fig. 9) and U_1U_2 (Fig. 10) are self-corresponding. Therefore U_2, V_1 (Fig. 9) and u_2, v_1 (Fig. 10) are coincident, and all we have proved is that u_4 passes through *one* fixed point, viz. the intersection of the ranges, and that U_4 lies on *one* fixed line, viz. the join of the vertices of the pencils.

In the case of perspective ranges and pencils, however, a direct proof of the existence of cross-axis and cross-centre is easily given as follows:

I. For ranges. Let O be the pole of perspective, S the intersection of the ranges, A_1A_2, B_1B_2 two corresponding pairs. Then $A_1A_2B_2B_1$ are vertices of a complete quadrangle of which O, S, (A_1B_2, A_2B_1) are diagonal points. Hence, by the harmonic property of the complete quadrangle, SO and the line joining S to (A_1B_2, A_2B_1) are harmonically conjugate with regard to the bases of the two ranges. But SO and these bases are fixed lines. Hence the line joining S to (A_1B_2, A_2B_1) is a fixed line. Therefore (A_1B_2, A_2B_1) lies on a fixed line, which is the cross-axis.

II. For pencils. Let x be the axis of collineation, s the join of the vertices, a_1a_2, b_1b_2 two corresponding pairs. Then $a_1a_2b_2b_1$ are sides of a complete quadrilateral of which (a_1b_1, a_2b_2), (a_1a_2, b_1b_2), (a_1b_2, a_2b_1), i.e. s, x and (a_1b_2, a_2b_1), are the diagonals. U_1U_2 is harmonically divided by x and (a_1b_2, a_2b_1). But x meets U_1U_2 at a fixed point: U_1, U_2 are themselves fixed. Hence the fourth harmonic is also fixed and (a_1b_2, a_2b_1) passes through a fixed point on s. This is the cross-centre.

We will close the present chapter with the following two theorems on the triangle, which are of importance.

31. Ceva's theorem. If the straight lines joining the vertices A, B, C of a triangle to a point O of its plane meet the opposite sides at P, Q, R, then

$$\frac{BP}{PC} \cdot \frac{CQ}{QA} \cdot \frac{AR}{RB} = +1.$$

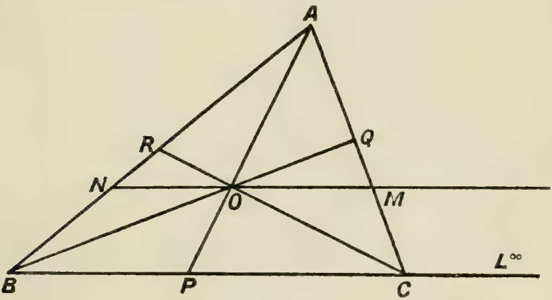


Fig. 12.

Through O draw a line LMN (Fig. 12) parallel to BC, meeting BC at L^∞, CA at M, AB at N. Then the ranges $(BPL^\infty C)$, $(QAMC)$ are perspective from O. Hence

$$\frac{BP \,.\, L^\infty C}{BC \,.\, L^\infty P} = \frac{BP}{BC} = \frac{QA \,.\, MC}{QC \,.\, MA} \quad\ldots\ldots\ldots\ldots\ldots (1).$$

Similarly $(CPL^\infty B)$, $(RANB)$ are perspective from O. Hence

$$\frac{CP \,.\, L^\infty B}{CB \,.\, L^\infty P} = \frac{CP}{CB} = \frac{RA \,.\, NB}{RB \,.\, NA} \quad\ldots\ldots\ldots\ldots\ldots (2).$$

Dividing (1) by (2),

$$\frac{BP}{PC} = \frac{QA}{CQ} \cdot \frac{CM}{MA} \cdot \frac{RB}{AR} \cdot \frac{AN}{NB},$$

or
$$\frac{BP}{PC}\cdot\frac{CQ}{QA}\cdot\frac{AR}{RB}=\frac{CM}{MA}\cdot\frac{AN}{NB}=1,$$
since MN being parallel to BC, $AN : NB = AM : MC$.

Note carefully that in the above the segments have to be taken with proper sign. The positive sense on each side of the triangle may be arbitrarily selected. It is usual to take it so that if we go round the triangle keeping the area on our left, we are moving in the positive sense throughout.

32. Menelaus' Theorem. If any straight line meet the sides BC, CA, AB of a triangle at P, Q, R respectively,

then
$$\frac{BP}{PC}\cdot\frac{CQ}{QA}\cdot\frac{AR}{RB}=-1.$$

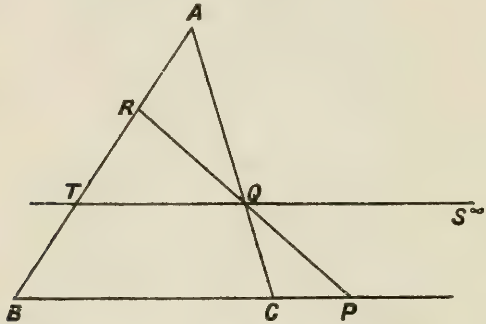


Fig. 13.

Through Q (Fig. 13) draw a parallel to BC meeting AB at T and BC at S^∞.

$(BPCS^\infty)$, $(BRAT)$ are perspective from Q.
$$\therefore \frac{BP\,.\,CS^\infty}{BS^\infty\,.\,CP}=\frac{BR\,.\,AT}{BT\,.\,AR}, \text{ or } \frac{BP}{CP}\cdot\frac{AR}{BR}=\frac{AT}{BT}=\frac{AQ}{CQ},$$
since TQ is parallel to BC.

Hence
$$\frac{BP}{CP}\cdot\frac{CQ}{AQ}\cdot\frac{AR}{BR}=+1,$$
or, reversing the signs of the three denominators,
$$\frac{BP}{PC}\cdot\frac{CQ}{QA}\cdot\frac{AR}{RB}=-1.$$

The theorems converse to those of Ceva and Menelaus are easily proved and are left as an exercise for the student.

EXAMPLES II A.

1. Show that the six cross-ratios of four points may be expressed in the form

$$\sin^2\theta,\ \cos^2\theta,\ \operatorname{cosec}^2\theta,\ \sec^2\theta,\ -\tan^2\theta,\ -\cot^2\theta.$$

2. Find all the cross-ratios of four harmonic points.

3. Prove that if I_1, J_2 be the vanishing points of two projective ranges, P_1, P_2 any pair of corresponding points, then

$$I_1P_1 \,.\, J_2P_2 = \text{constant}.$$

4. Give a geometrical construction connecting the points of two projective ranges when the vanishing points of the ranges and a pair of corresponding points are given.

5. Two similar coplanar ranges have a self-corresponding point. Show that the lines joining their corresponding points are all parallel.

6. If two similar ranges lie on parallel lines, the joins of corresponding points pass through a fixed point.

7. Through the points of one of two coplanar similar ranges lines are drawn parallel to a given direction in the plane and through the corresponding points of the other range lines are drawn parallel to another given direction in the plane. Show that the intersections of corresponding lines lie on a fixed straight line.

[The points at infinity correspond. Take vertices S_1, S_2 of Art. 26 on line at infinity and result follows.]

8. All the vertices but one of a polygon lie on fixed lines, while its sides are parallel to fixed directions. Show that the locus of the last vertex is a straight line.

9. Two collinear projective ranges are given by two corresponding triads $A_1B_1C_1$, $A_2B_2C_2$. Give completely a geometrical construction to find the point P_2 of the second range corresponding to a given point P_1 of the first.

10. Two concentric projective pencils are given by two corresponding triads $a_1b_1c_1$, $a_2b_2c_2$. Give completely a geometrical construction to find the ray p_2 of the second pencil corresponding to a given ray p_1 of the first.

11. Two concentric pencils are *oppositely* equal. Show that the two bisectors of the angles between any two corresponding rays are self-corresponding.

12. A, B are two fixed points: P_1, P_2 are harmonically conjugate with regard to A, B. Show that the ranges $[P_1]$, $[P_2]$ are projective and find a geometrical construction by projections to pass from one to the other. What are the correspondents of the points A, B?

13. Show that if $\{APBQ\} = \{AP'BQ'\}$, then $\{APBP'\} = \{AQBQ'\}$. Deduce that if A, B are self-corresponding elements of two collinear projective ranges, any two corresponding points determine with A, B a constant cross-ratio.

14. Prove that if

$$\{A_1 B_1 C_1 P_1\} = \{A_2 B_2 C_2 P_2\}$$
$$\{A_1 B_1 C_1 Q_1\} = \{A_2 B_2 C_2 Q_2\}$$
$$\{A_1 B_1 C_1 R_1\} = \{A_2 B_2 C_2 R_2\}$$
$$\{A_1 B_1 C_1 S_1\} = \{A_2 B_2 C_2 S_2\}$$

then $$\{P_1 Q_1 R_1 S_1\} = \{P_2 Q_2 R_2 S_2\}.$$

15. Prove that if two corresponding ranges be such that any four elements of one have the same cross-ratio as the corresponding four elements of the other they are projective.

16. If EFG be the diagonal triangle of a complete quadrangle $ABCD$ and the sides of EFG meet the sides of the quadrangle in I, J, K, L, M, N, show that I, J, K, L, M, N are the six vertices of a complete quadrilateral having for its diagonal triangle EFG.

17. If efg be the diagonal triangle of a complete quadrilateral $abcd$ and the vertices of efg be joined to the vertices of the quadrilateral by lines i, j, k, l, m, n, show that i, j, k, l, m, n are the six sides of a complete quadrangle having efg for its diagonal triangle.

18. Given the cross-axis of two projective ranges and a pair of corresponding points, show how to construct the point of one range corresponding to a given point of the other. In particular construct the vanishing points.

19. Given the cross-centre of two projective pencils and a pair of corresponding rays, find a construction for the ray of one pencil corresponding to a given ray of the other.

20. A ray through a fixed point O cuts a line u at P_1 and the line at infinity at P_2^∞. P_1, P_2^∞ then describe projective ranges on u and on the line at infinity respectively. Show that the cross-axis of these two ranges is a parallel to u at a distance from u equal to the distance of O from u.

21. The arms OP, OQ of an angle of fixed magnitude which moves in one plane about its fixed vertex O intersect two given straight lines at P and Q respectively. Show that the ranges $[P]$, $[Q]$ are projective.

22. If in Ex. 21 one of the given straight lines is the line at infinity, construct the cross-axis of the ranges $[P]$, $[Q^\infty]$.

23. Through a point O a ray OPQ is drawn meeting two fixed lines at P, Q. If R be harmonically conjugate to O with regard to P, Q prove that the locus of R is a straight line.

24. A, B are two fixed points, u a fixed line. If P be any point of u and p be harmonically conjugate to u with regard to PA, PB, show that p passes through a fixed point.

25. If the vertices of a polygon lie on fixed concurrent lines, while all the sides but one pass through fixed points, the last side also passes through a fixed point.

26. If the sides of a polygon pass through fixed collinear points, while all the vertices but one move on fixed straight lines, the locus of the last remaining vertex is a straight line.

27. Prove the converse of Menelaus' and Ceva's Theorems.

[Exs. 28—36 follow from the theorems of Menelaus and Ceva.]

28. Prove that the three medians of a triangle meet at a point.

29. Prove that the three perpendiculars from the vertices of a triangle on the opposite sides meet at a point.

30. If ABC be a triangle, D the mid-point of BC, and if AD' be a line making with AB, AC the same angles that the median AD makes with AC, AB, then AD is called the *symmedian* through A. Show that the three symmedians meet at a point.

31. Apply Menelaus' Theorem to prove Desargues' Theorem that if ABC, $A'B'C'$ be two coplanar triangles such that AA', BB', CC' are concurrent, then aa', bb', cc' are collinear and conversely.

32. ABC is a triangle, O any point in its plane. If OA meet BC at P, OB meet CA at Q, OC meet AB at R, and if P' be the harmonic conjugate of P with regard to BC, Q' the harmonic conjugate of Q with regard to CA, R' the harmonic conjugate of R with regard to AB, show that P', Q', R' are collinear.

33. In Ex. 32 prove that the middle points of PP', QQ', RR' are collinear. Hence prove that the middle points of the diagonals of a quadrilateral are collinear.

34. The vertices of a triangle are joined to the points of contact of the opposite sides with one of the escribed circles. Show that the lines thus formed are concurrent.

35. Pairs of points P and P', Q and Q', R and R' are taken on the sides BC, CA, AB of a triangle, such that AP and AP', BQ and BQ', CR and CR' are equally inclined to the bisectors of the angles A, B, C. Prove that if AP, BQ, CR are concurrent, then so also are AP', BQ', CR'.

36. Pairs of points P and P', Q and Q', R and R' are taken on the sides BC, CA, AB of a triangle and equidistant from their mid-points. Show that if AP, BQ, CR are concurrent, then so also are AP', BQ', CR'.

EXAMPLES II B.

1. $A_1B_1C_1D_1$, $A_2B_2C_2$ are given by distances from a fixed origin O equal to 2, 1, -3, 4; -1, 5, 2 respectively.

Construct geometrically a point D_2 such that

$$\{A_2B_2C_2D_2\}=\{A_1B_1C_1D_1\}$$

and verify your result by calculation.

2. $O=(1, 0)$; $O'=(-1, 0)$; $A=(2, 3)$; $B=(\cdot 5, 2\cdot 5)$; $C=(-\cdot 5, 1)$; $D=(0, 4)$. Construct a ray $O'D'$ such that $O\{ABCD\}=O'\{ABCD'\}$.

3. Construct the locus of cross-joins of the ranges defined by the triads (0, 0), (0, 2), (0, 1); (1, 0), (0, 0), (3, 0) respectively, the axes of co-ordinates being inclined at 75°. Hence construct any pair of corresponding points of the ranges and the envelope of the joins of such points.

4. A, B, C are three points of a straight line, $AB=2$, $BC=1$. Construct points P, Q, R which shall be harmonically conjugate to A with respect to BC, B with respect to CA, C with respect to AB.

5. Construct a ray OD harmonically conjugate to OB with regard to OA, OC where the angles AOB, BOC are 30° and 15° respectively.

6. Using the ruler only, draw a line through a given point P and the inaccessible meet Q of two straight lines a, b.

CHAPTER III.

PROJECTIVE PROPERTIES OF THE CONIC.

33. Definition of the Conic. A *conic section* or *conic* is the projection of a circle, or the plane section of a cone (right or oblique) on a circular base.

Since in general a straight line meets a circle in two points, the same is true of a conic, because properties of incidence are unaltered by projection: and since from any point two tangents can in general be drawn to a circle, the same holds for the conic since properties of tangency are unaltered by projection.

It follows from the definition that any property of the circle which is projective, i.e. unaltered by projection, can be transferred at once to the conic.

34. Types of Conic. There are three types of conic, according as in the original figure the vanishing line cuts the circle in two real distinct points, or in two real coincident points (i.e. touches it) or does not cut it in real points.

In the first case there are two distinct points at infinity on the conic, namely the points I_2^∞, J_2^∞ corresponding to the intersections I_1, J_1 of the circle with the vanishing line. Such a conic is called a *hyperbola*.

The tangents to the circle at I_1, J_1 project into the tangents at I_2^∞, J_2^∞ to the conic. These two tangents are called the *asymptotes* of the conic. The curve has two branches, corresponding to the two parts into which the vanishing line divides the circle, cf. Fig. 6.

If the vanishing line touch the circle I_1, J_1 coincide.

The conic has two coincident points at infinity, i.e. it has the line at infinity for one of its tangents. Such a conic is called a *parabola*. It consists of one branch extending to infinity.

If the vanishing line do not cut the circle in real points there are no real points at infinity on the conic. The conic

consists of an oval lying entirely at a finite distance and is called an *ellipse*.

Since in cylindrical projection the vanishing line is at infinity the ellipse is the only one of the conics which can be obtained from the circle by cylindrical projection.

There are also two other types of conic, viz. the *line-pair* and *point-pair*. These are to some extent anomalous, and will be discussed in Art. 44.

35. Notation for projective ranges and pencils. Curve as envelope and locus. To abridge proofs, the words "is projective with" will in future (except in enunciations) be denoted by the symbol $\overline{\wedge}$. Thus $[P_1] \overline{\wedge} [P_2]$ is to be read: the range described by P_1 is projective with the range described by P_2.

The terms *locus* and *envelope* will frequently occur in what follows. A curve may be generated in two ways: (a) by a moving point P; we then speak of the curve as the locus of P and we construct it graphically from a large number of positions of P, forming a closely inscribed polygon; (b) by a moving tangent p; we then speak of the curve as the *envelope* of p and we construct it graphically from a large number of positions of p, forming a closely circumscribed polygon.

36. Chasles' Theorem. If P be a variable point on a circle, p the tangent at P, O any fixed point on the circle, t any fixed tangent to the circle, then the pencil $O[P]$ is *equi-anharmonic* with the range $t[p]$, that is, if P, Q, R, S be any four positions of P, p, q, r, s the corresponding tangents, then

$$O\{PQRS\} = t\{pqrs\}.$$

Let C (Fig. 14) be the centre of the circle, T the point of contact of t, $P' = pt$. Then $P'P$, $P'T$ being tangents to a circle, the angle $P'CT = \frac{1}{2} PCT =$ angle at the circumference POT.

Therefore by placing O on C and OT on CT the pencils $O[P]$, $C[P']$ are superposable. Hence they are directly equal. Hence by Art. 24

$$O[P] \overline{\wedge} C[P'],$$

$$\therefore O\{PQRS\} = C\{P'Q'R'S'\} = \{P'Q'R'S'\} = t\{pqrs\}.$$

37. Pencils obtained by joining a variable point to two fixed points. If O' be any other fixed point on the circle, we have by Chasles' theorem

$$O'[P] \overline{\wedge} C[P'] \overline{\wedge} O[P].$$

Hence the joins of a variable point P on a circle to two fixed points O, O' on the circle sweep out projective pencils.

In the case of the circle these pencils are clearly *directly equal*, for the angle TOP = angle $TO'P$ (Fig. 14), by the well-known property of angles in the same segment.

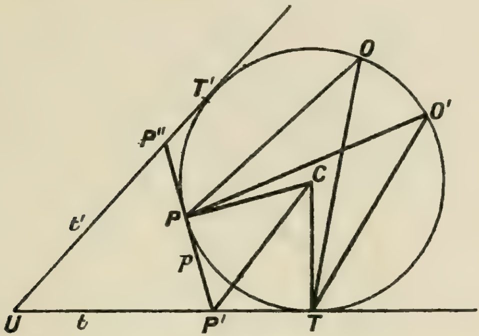


Fig. 14.

38. Ranges obtained by intersections of a variable tangent and two fixed tangents.

Let t' (Fig. 14) be any other fixed tangent. Let $pt' = P''$. Then by Chasles' Theorem

$$C[P''] \,\overline{\wedge}\, O[P] \,\overline{\wedge}\, C[P'].$$

Cutting the projective pencils $C[P']$, $C[P'']$ by t, t'

$$[P'] \,\overline{\wedge}\, [P''],$$

or the intersections of a variable tangent p to a circle and two fixed tangents t, t' describe two projective ranges.

39. Corresponding properties for the conic. Since cross-ratios are not altered by projection and projective ranges and pencils project into projective ranges and pencils respectively, the properties stated in Arts. 36—38 hold for the conic, except that now the pencils will no longer be *equal*, for equal angles are not, in general, projected into equal angles. But the properties

$$O\{PQRS\} = t\{pqrs\} \quad \text{.......(1)},$$

$$O[P] \,\overline{\wedge}\, O'[P] \quad \text{.......(2)},$$

$$t[p] \,\overline{\wedge}\, t'[p] \quad \text{.......(3)},$$

hold equally if for the word "circle" in the last three articles we read "conic."

From the property (2) it follows that every conic may be

obtained as the locus of meets of corresponding rays of two projective pencils. From the property (3) it follows that every conic can be obtained as the envelope of joins of corresponding points of two projective ranges.

If in Fig. 14 P approaches O', OP approaches OO', $O'P$ approaches the tangent at O. Hence to OO' considered as a ray of the pencil O corresponds the tangent at O'. Similarly to $O'O$ considered as a ray of the pencil O' corresponds the tangent at O. The cross-centre (Art. 30) of the two pencils through O, O' is therefore the point of intersection of the tangents at O, O'.

Again, if p approaches t, P' approaches T and P'' approaches the intersection U of the two tangents t, t'. Hence to tt' considered as a point of range t' corresponds the point of contact T of t. Similarly to tt' considered as a point of range t corresponds the point of contact T' of t'. The cross-axis of the two ranges is therefore the chord of contact TT' (Art. 30).

In the above reasoning it is immaterial whether the curve of Fig. 14 be a circle or a conic.

40. Property of tangents to a parabola. The property (3) of the last article takes a particularly simple form when the conic is a parabola. For then the line at infinity is a tangent to the curve by Art. 34. Hence the points at infinity of the ranges t, t' correspond, and by Art. 24 the ranges are similar. But T, P', U correspond to U, P'', T': hence

$$TP' : P'U = UP'' : P''T',$$

or the intercepts made by a variable tangent to a parabola on two fixed tangents are inversely proportional. This furnishes an easy graphical method of drawing a parabola as an envelope, two tangents UT, UT' and their points of contact T, T' being given. n being a large integer, take lengths $\frac{1}{n}UT'$ and $\frac{1}{n}UT$ and lay them off in succession any number of times upon UT', TU respectively, starting from U along UT' and from T along TU. Join corresponding points of division; each of these is a tangent to the parabola.

41. The product of any two projective pencils is a conic. We shall call the locus of meets of corresponding rays of two pencils the *product* of the pencils and the envelope of joins of corresponding points of two ranges the *product* of the ranges.

We have seen that every conic can be obtained as the product of two projective pencils. But these pencils might be projective pencils of a *special* type (as in the case of the circle, where they

are *equal*). We will now show that any two projective pencils whatever lead to a conic locus.

Let $S[P]$, $O[P]$ (Fig. 15) be the two pencils and let OT be the ray of the pencil O corresponding to SO of the pencil S. Draw any circle touching OT at O. Let this circle meet OP at P', OS at S'. By Art. 37

$$O[P'] \,\overline{\wedge}\, S'[P'].$$

But
$$O[P'] = O[P] \,\overline{\wedge}\, S[P],$$
$$\therefore S[P] \,\overline{\wedge}\, S'[P'].$$

Also $S'O$ of pencil S' corresponds to the tangent at O of

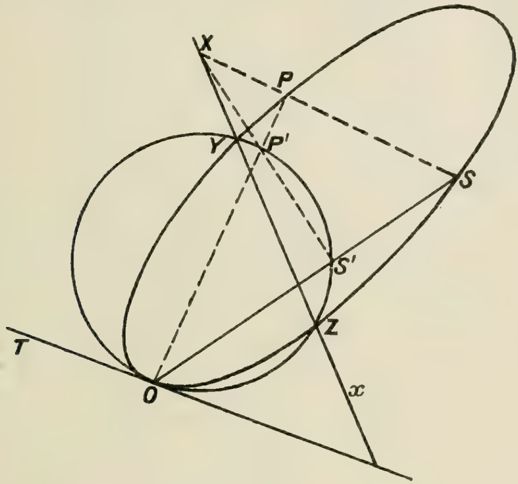


Fig. 15.

pencil O, i.e. to OT, and this in turn corresponds to SO. Hence in pencils $S[P]$, $S'[P']$ the ray SS' is self-corresponding. Therefore by Art. 26 $S[P]$, $S'[P']$ are perspective, therefore corresponding rays SP, $S'P'$ meet at X on a fixed line x. P is therefore constructed from P' by the construction for two figures in plane perspective, O being the pole of perspective, x the axis of collineation, S and S' a given pair of corresponding points. For PP' passes through O and SP, $S'P'$ meet on x.

The locus of P is thus in plane perspective with the circle which is the locus of P', that is, it may be looked upon as

the rabatted projection of this circle upon another plane. It is therefore a conic by definition. Note that the conic and circle touch at O; if they intersect again at Y, Z, then Y, Z must be self-corresponding points and the axis of collineation x passes through them.

42. The product of any two projective ranges is a conic. Let $[P]$, $[P_0]$ (Fig. 16) be the ranges, t, x their bases, $p = PP_0$. Let T be the point of range x corresponding to the intersection U of tx. Draw any circle touching x at T and from

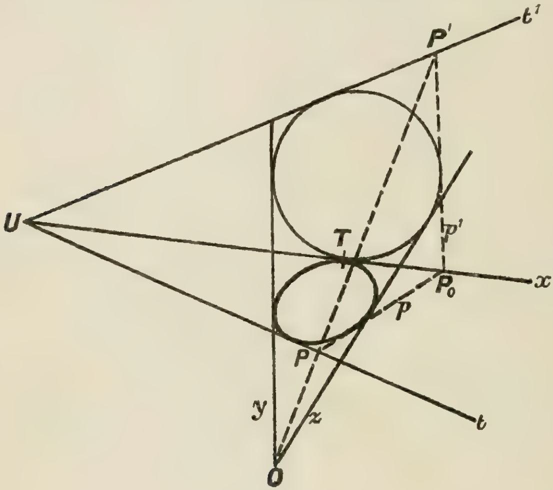


Fig. 16.

U, P_0 draw tangents t', p' to this circle. Let $p't' = P'$. Then $[P'] \overline{\wedge} [P_0] \overline{\wedge} [P]$. Also U of range t' corresponds to T of range x and T of range x corresponds to U of range t. Hence the ranges t, t'—or $[P]$, $[P']$—have a self-corresponding point U. They are therefore perspective ranges by Art. 26. Hence PP' passes through a fixed point O. The lines p', p are obtained from one another by the construction for figures in plane perspective, O being the pole, x the axis, t, t' a pair of given corresponding lines. For p, p' meet on x; $(pt, p't')$ passes through O. Hence the envelope of p is in plane perspective with the envelope of p' and, as in the last article, must be a conic.

The conic and circle both touch x at T. If they have other real common tangents y, z, these must be self-corresponding lines and so pass through O.

43. Deductions from the above. In the proofs of Arts. 41, 42 the circle may clearly be replaced by any conic. For the only properties of the circle made use of in the proofs are also, by Art. 39, properties of the conic.

It follows that two conics in contact can be brought into plane perspective in two ways, viz. (1) by taking the point of contact to be the pole of perspective: the axis of collineation is then a line passing through the remaining two intersections of the two conics; (2) by taking the common tangent to be the axis of collineation: the pole of perspective is then a point through which pass the remaining common tangents of the two conics.

These, however, are not the only ways in which such conics can be brought into plane perspective (cf. Exs. III A. 14).

Note also that the product of two projective pencils passes through the vertices of the pencils.

Thus the product of two projective pencils of parallel rays is a hyperbola whose points at infinity and therefore the directions of whose asymptotes are given by the directions of the two pencils.

Similarly the product of two projective ranges touches the bases of the ranges. Thus the product of a range at a finite distance and a projective range on the line at infinity is a parabola.

If therefore through a fixed point O a ray OP be drawn meeting a fixed line u at P, and PQ^∞ be drawn through P making a fixed angle α with OP, PQ^∞ touches a fixed parabola. For draw OQ^∞ parallel to PQ^∞, the pencils $O[P]$, $O[Q^\infty]$ are superposable by means of a rotation α about O. They are therefore equal and projective. Hence the ranges $[P]$, $[Q^\infty]$ are projective. The latter being on the line at infinity, the result stated follows.

It will also follow from Art. 41, since projective pencils project into projective pencils, that the projection of a conic is a conic.

44. Line-pair and point-pair. If the two projective pencils of Art. 41 are *perspective* their product breaks up into two straight lines, namely the axis of collineation and the self-corresponding ray, since the latter may be regarded as intersecting itself at any one of its points, and therefore must figure in the locus of intersection of corresponding rays.

A *line-pair* is therefore a special case of a conic locus.

If the two projective ranges of Art. 42 are perspective, their product breaks up into two sets of lines, one passing through the vertex of perspective, the other passing through the self-corresponding point, since any ray through the latter may be looked upon as the join of the point to itself. The envelope then reduces to these two points, so that:

A *point-pair* is a special case of a conic envelope.

The line-pair and point-pair present certain anomalies which should be noticed.

Let the components of a line-pair be a, b and their meet C. Then a line through a point P of the plane meets the line-pair in two distinct points, unless the line passes through C when the intersections coincide. Thus from any point one tangent, and one only, can be drawn to a line-pair. We may, to keep the properties of conics perfectly general, look upon these as two coincident tangents, but it is then no longer true that a point, the two tangents from which to a conic are coincident, is itself on the conic.

On the other hand, let the components of a point-pair be A, B and their join c. From a point P two distinct tangents PA, PB can be drawn to the point-pair, except if P be on c when they coincide. The points of c may therefore be looked upon as belonging to the point-pair conic. Any straight line then meets the point-pair in one point, and one only. If we look upon this point as two coincident points, to preserve the property that a straight line meets a conic in two points, it will appear that a straight line can meet a conic in two coincident points, without being a tangent to it, for lines not through A and B are not tangents to the point-pair.

The true significance of the line-pair and point-pair will be more apparent later on when we come to study the central and focal properties of the conics.

It is interesting to note the manner in which the line-pair and point-pair appear as projections of a circle. To obtain the line-pair, project the circle from a vertex V outside its plane and cut the cone so formed by a plane through V, i.e. take the vertex *in* the plane of projection. To obtain the point-pair take V in the plane of the circle and project on to any other plane.

45. A conic is determined by five points or by five tangents. For let O, O', A, B, C be five points on a conic, P any sixth point, then the pencils

$$O(ABCP), \quad O'(ABCP)$$

are projective. But O, O', A, B, C determine completely the corresponding triads $O(ABC)$, $O'(ABC)$, and these in turn determine completely the relation between the pencils. Hence OP being given, $O'P$ is known, that is, P is determined. Every point on the conic is therefore fixed when five points are fixed. It follows that two distinct conics cannot have more than four points of intersection.

The points A, O (and also the points B, O') may coincide without making the constructions indeterminate, provided we interpret OA, $O'B$ as the tangents at O, O'. Accordingly being given a point on the conic and the tangent at this point is equivalent to being given two points.

Similarly if t, t', a, b, c be five tangents to a conic, p any sixth tangent, pt, pt' are corresponding points of the projective ranges defined by the triads $t(abc)$, $t'(abc)$. Therefore when pt is known, pt' is known, and p is determined. Thus every tangent is determinate when five are given. It follows that two distinct conics cannot have more than four common tangents.

As before a, t (and also b, t') may coincide without making the constructions indeterminate, provided we interpret at, bt' as the points of contact of t, t'. Thus being given a tangent and its point of contact is equivalent to being given two tangents.

46. Construction for a conic through five points. Take two of the given points for vertices of two projective pencils and obtain corresponding triads by joining to the three remaining points. Construct pairs of corresponding rays by the method of Art. 26, or any other. The intersections of corresponding rays are points on the conic.

A precisely similar method can be applied to obtain a conic as an envelope when five tangents are given.

EXAMPLES III A.

1. Show that with four points A, B, C, D on a conic may be associated a definite cross-ratio: and also that with four tangents a, b, c, d may be associated a definite cross-ratio; and show that the cross-ratio of four such tangents is the cross-ratio of their four points of contact.

2. O, O', A, B, C, D are six points on a conic. If $(OA, O'B) = P$, $(OB, O'C) = Q$, $(OC, O'D) = R$, $(OD, O'A) = S$, prove that if P, Q, R, S, O, O

lie on a conic the rays OB, OD are harmonically conjugate with regard to OA, OC.

3. A variable tangent LL' meets two fixed parallel tangents to a conic whose points of contact are A and A' at L and L'; prove that the rectangle $AL \,.\, A'L'$ is constant.

4. A variable tangent meets the asymptotes of a hyperbola at P, P'. If C be the intersection of the asymptotes, prove that $CP \,.\, CP' = $ const.

5. On the tangent at O to a conic any point P is taken and PT is drawn to touch the conic at T. If S be any other fixed point on the conic, show that the locus of the intersection of OT, SP is another conic, which touches the original conic at O and S.

6. O, O' are two fixed points on a conic s; l is a fixed straight line. P is any point on s; OP, $O'P$ are joined, meeting l at R, R' respectively; OR', $O'R$ meet at Q. Show that the locus of Q is another conic.

7. The arms OA, OB of an angle of fixed magnitude a meet a fixed straight line at A and B, and through A and B respectively lines AP, BP are drawn parallel to fixed directions. Show that the locus of P is a hyperbola and find its asymptotes.

8. Two oppositely equal pencils have two different vertices. Show that their product is a rectangular hyperbola (i.e. one whose asymptotes are at right angles). [Find when corresponding rays of the two pencils are parallel.]

9. O, S are fixed points, a, b fixed straight lines. A line through O meets a at P, b at Q. Through Q is drawn a parallel to SP. Show that this parallel touches a fixed parabola.

10. OT, OV are tangents to a parabola whose points of contact are T and V. Show that the tangent to the parabola parallel to TV bisects OT, OV.

11. Two circles touch one another. From a point on their common tangent lines are drawn touching the circles at P, P'. Show that PP' passes through a fixed point.

12. Two circles touch one another. From their point of contact a ray is drawn meeting the circles at P, P'. Show that tangents at P, P' are parallel.

13. Given three points on a hyperbola and the directions of its asymptotes, show how to construct the pencils of parallel rays which generate the hyperbola and deduce a construction for the asymptotes.

14. Show that any two conics s_1, s_2 in a plane can be brought into a plane perspective relation by taking one of their common chords as axis

of collineation. [Let XY be the common chord, A_1A_2 a common tangent touching s_1 at A_1, s_2 at A_2; Z a point on XY: let A_1Z meet s_1 at B_1, A_2Z meet s_2 at B_2. Take $O=(B_1B_2, A_1A_2)$; then the perspective relation defined by pole O, axis XY and pair of corresponding points A_1, A_2 transforms s_1 into a conic through the three points X, Y, B_2, and touching OA_2 at A_2, i.e. into s_2.]

15. From any point on one of their common chords tangents are drawn to two conics, touching the conics at P, Q. Show that PQ passes through one of two fixed points.

16. Show that any two conics s_1, s_2 in a plane can be brought into a plane perspective relation by taking the meet of two of their common tangents as pole of perspective.

17. Through the meet O of the common tangents to two conics a line is drawn meeting one conic at P and the other at Q. Show that the tangents at P, Q meet on one of two fixed lines.

18. A tangent to a conic at P meets a fixed tangent at Q and QR is drawn through Q parallel to OP where O is a fixed point on the conic. Show that QR touches a parabola.

19. The sides of a polygon pass through fixed points and all the vertices but one lie on fixed lines. What is the locus of the last remaining vertex?

20. The vertices of a polygon lie on fixed lines and all the sides but one pass through fixed points. What is the envelope of the last remaining side?

EXAMPLES III B.

1. There are two projective pencils of rays whose centres are at the points (1, 2) and (4, 6). The rays

$$y-2=0,\ 2x-y=0,\ x-y+1=0$$

of the first pencil correspond to the rays

$$x-y+2=0,\ 2x-3y+10=0,\ x-4=0$$

respectively of the second pencil.

Construct the ray of the second pencil corresponding to the ray $4x-3y+2$ of the first pencil and the conic which is the locus of the intersections of corresponding rays of the pencils. Construct the tangents to this conic at the points (1, 2) and (4, 6).

2. AB, AC are two lines inclined at 60°. $AB=2''$; $AC=4''$. Construct by tangents the parabola which touches AB, AC at B and C.

3. Two projective pencils of parallel rays are given by the triads $x=0, 1, 3$, $y=0, -2, 1$, the axes being inclined at $60°$. Draw the locus of intersections of corresponding rays of the pencils and construct its asymptotes.

4. Draw any two circles in contact. Verify that if tangents be drawn to the circles from any point on their common tangent, the join of the points of contact passes through a fixed point.

CHAPTER IV.

POLE AND POLAR.

47. Polar of a point with regard to a circle. Let O be any point in the plane of a circle and let a variable ray OPQ be drawn to meet the circle at P, Q. To prove that the locus of points R harmonically conjugate to O with regard to P, Q is a straight line.

Let the diameter through O (Fig. 17) meet the circle at

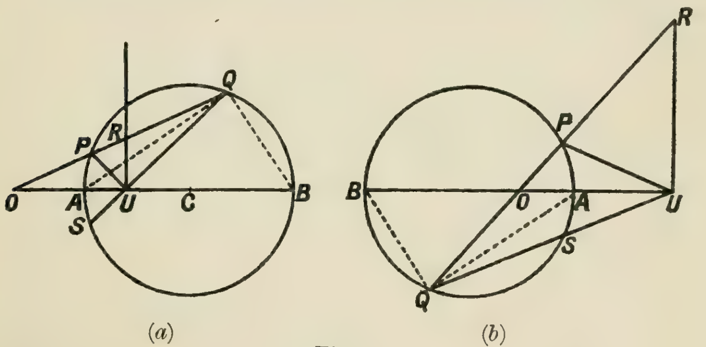


Fig. 17.

A, B. Let S be the point on the circle symmetrical with P with regard to AB. Join SQ meeting AB at U. Then because arc AS = arc AP, the angle AQS = angle AQO. Hence QA, and its perpendicular QB, are the internal and external bisectors of the angle UQO. Hence by Art. 28, $Q(OAUB)$ is a harmonic pencil, $\therefore (OAUB)$ is a harmonic range. O, A, B being fixed points, U is a fixed point.

Also UO and the perpendicular to UO through U are the internal and external bisectors of the angle PUS, for by sym-

metry the angles PUO, SUO are equal. Hence if this perpendicular meet OPQ at R, the pencil $U(OPRQ)$, and $\therefore$ the range $(OPRQ)$ is harmonic. The locus of R is therefore a fixed line, namely the perpendicular through U to OU.

This locus is called the *polar* of O with regard to the circle, conversely O is called the *pole* of UR with regard to the circle. Given UR, its pole with regard to the circle is a well determined point, being that point on the diameter perpendicular to UR which together with UR divides that diameter harmonically.

48. Polar as chord of contact. If real tangents can be drawn from O to the circle, their points of contact lie on the polar of O.

For if the conjugate points P, Q coincide R must coincide with them by Art. 28.

Hence the polar of O is the chord of contact of tangents from O to the circle.

49. Relation between distances of a point and of its polar from the centre of the circle. Let C (Fig. 17 a) be the centre of the circle.

Then by Art. 28, C being halfway between the two points A, B, which are harmonically conjugate with regard to O, U

$$CU \,.\, CO = CA^2 = \text{square of radius of circle.}$$

50. Polar of a point on the circle. If O be at A, $CU.CA = CA^2$ gives $CU = CA$, or U is at A. Hence UR is perpendicular to the radius at A, that is, it is the tangent at A. Thus the polar of a point on the circle is the tangent at the point.

Conversely the pole of a tangent to the circle is its point of contact.

51. Conjugate points and lines. If a point P lies on the polar of Q with regard to a circle, then Q lies on the polar of P.

If the line PQ cut the circle in real points S, T the proof is obvious. For $\because$ P is on the polar of Q $\therefore$ P, Q are harmonically conjugate with regard to S, T; but this is the condition that Q should lie on the polar of P.

But if PQ do not cut the circle in real points, let O (Fig. 18) be the centre of the circle. Let the polar of Q, PQ' meet OQ at Q'. PQ' is perpendicular to OQ, $\therefore$ circle on PQ as diameter passes through Q'. Let this circle meet OP again at P'. Then QP' is perpendicular to OP and $OP \,.\, OP' = OQ \,.\, OQ' =$ square

of radius (since Q is pole of PQ'). Hence $P'Q$ is the polar of P, or the polar of P passes through Q.

Two points which have the property that the polar of either passes through the other are said to be *conjugate points with regard to the circle.*

Calling p the polar of P, q the polar of Q, the theorem stated at the beginning of this article may be written : if the

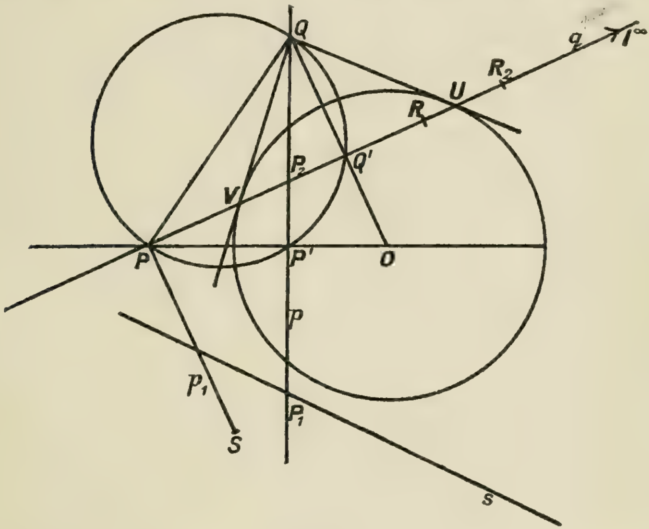


Fig. 18.

pole of p with regard to a circle lies on q, then the pole of q lies on p.

Two such lines p, q are said to be *conjugate lines with regard to the circle.*

52. Circle on line joining two conjugate points as diameter. Referring to Fig. 18, since $OP \,.\, OP' =$ square of tangent from O to circle on PQ as diameter $=$ square of radius of original circle, the radius of the original circle is equal to the tangent from O to the circle on PQ as diameter ; the two circles therefore cut at right angles. Hence the circle on the line joining two points conjugate with regard to a given circle is orthogonal to the given circle.

Conversely if any circle be orthogonal to the given circle, the extremities PQ of any diameter are conjugate points with regard to the given circle. For let O be the centre of given circle; join OQ, OP meeting the orthogonal circle again at Q', P'.

Then $OQ \, . \, OQ'$ = square of tangent to orthogonal circle
= square of radius of given circle.

$\therefore$ PQ' (being perpendicular to OQ) is the polar of Q. Hence P, Q are conjugate points.

If PQ meet the given circle at S, T, then S, T are harmonically conjugate with regard to PQ.

Hence if two circles are orthogonal, every diameter of either which meets the other in real points is harmonically divided by that other.

53. **Two conjugate lines through a point are harmonically conjugate with regard to the two tangents from the point.** For if from Q (Fig. 18) two tangents QU, QV be drawn to the circle, UV is the polar of Q. If any line through Q meet UV at P_2, the pole P of QP_2 is on UV. QP, QP_2 are conjugate lines through Q. But QP_2 being the polar of P with regard to the circle P, V, P_2, U are four harmonic points. Therefore $Q\,(PVP_2U)$ is a harmonic pencil, which proves what was required.

54. **The polars of a range form a pencil equi-anharmonic with the range and conversely.** Consider points P on a line q (Fig. 18). Let p be the polar of P with regard to the circle whose centre is O, Q the pole of q with regard to this circle. Then p passes through Q. Hence the lines p form a pencil. Also OP is perpendicular to p. Hence by a translation of O to Q and rotation through a right angle the pencils $O\,[P]$ and $[p]$ are superposable.

$$\therefore \quad [p] \,\overline{\wedge}\, O\,[P].$$

But $[P]$ is a section of $O\,[P]$ and $\therefore$ equi-anharmonic with $O\,[P]$, $\therefore$ $[P]$ is equi-anharmonic with $[p]$.

Conversely it may be shown that when p describes a pencil with vertex Q, its pole P describes an equi-anharmonic range on q.

55. **Conjugate ranges and pencils are projective.** To any point P there is one conjugate point P_1 on any line s and one only, namely the point where the polar p of P cuts s (Fig. 18). But by the last article the pencil $[p] \,\overline{\wedge}\, O\,[P]$. Hence taking sections by q, s, we have $[P] \,\overline{\wedge}\, [P_1]$. Thus the ranges described

on two given lines by points conjugate with regard to a given circle are projective. Such ranges may be called conjugate ranges with regard to the circle.

In like manner there exists one line conjugate to any line p through any given point S and one only, namely the line p_1 joining S to the pole P of p. Let p turn about Q, then P moves on q: we have $[p] \barwedge O[P]$. But $S[P] \barwedge O[P]$, these pencils being perspective. Hence $[p] \barwedge S[P] \barwedge [p_1]$.

Therefore the pencils described by conjugate lines through two given points are projective. They may be called conjugate pencils with regard to the circle.

The conjugate ranges may be on the same line. Thus if P be any point of q and the polar p of P meet q at P_2 (Fig. 18), then P, P_2 are conjugate points on q and $[P] \barwedge [P_2]$.

Similarly conjugate pencils may be concentric. Thus to p through Q, QP is conjugate and $Q[P] \barwedge [p]$.

Note that since when a point is on the circle, it lies on its own polar, the points where a line cuts the circle are self-corresponding points of the conjugate ranges on that line, and in a precisely similar manner the two tangents to the circle from a point are self-corresponding rays of the conjugate pencils through that point.

56. Conjugate ranges on a tangent and conjugate pencils through a point on the circle. An important exceptional case occurs when a line is a tangent to the circle. In this case the polar of any point on it other than the point of contact passes through the point of contact. The point of contact is therefore conjugate to every point on the tangent and conversely.

In like manner if we have a point on the curve, the pole of any ray through the point lies on the tangent at the point. The tangent at the point is therefore conjugate to every other line through the point and conversely.

57. Polar of centre. Since chords through the centre are bisected at the centre, the points harmonically conjugate to the centre on these chords are at infinity by Art. 28. The polar of the centre is therefore the line at infinity.

Conversely the polar of a point at infinity, say P^∞, passes through the centre, i.e. is a diameter. Also this polar is the chord of contact of the two tangents parallel to the direction of P^∞. But this chord of contact, being a diameter, is perpendicular to the tangents at its extremities. Hence the polar of P^∞ is the diameter perpendicular to the direction of P^∞. The

line through the centre conjugate to this diameter passes through P^∞, i.e. is perpendicular to this diameter. Therefore conjugate diameters of a circle are at right angles.

58. Relation between distances of conjugate points on a line. Consider the point I^∞ at infinity on q (Fig. 18). Its polar is the diameter OQ perpendicular to q. Therefore its conjugate point on q is Q'. Let R, R_2 be any other pair of conjugate points on q, and note that if I^∞ is conjugate to Q', so is Q' conjugate to I^∞. Thus in the conjugate ranges on q the set of four $PQ'RI^\infty$ corresponds to the set of four $P_2I^\infty R_2Q'$. Equating the cross-ratios,

$$\frac{PQ'.RI^\infty}{PI^\infty.RQ'}=\frac{P_2I^\infty.R_2Q'}{P_2Q'.R_2I^\infty},$$

i.e.
$$\frac{PQ'}{RQ'}=\frac{R_2Q'}{P_2Q'},$$

or
$$Q'P.Q'P_2=Q'R.Q'R_2.$$

Hence the product of the distances of conjugate points from the foot of the perpendicular from the centre on the given line is constant.

To find this constant, consider a point U where q meets the circle. U is a self-corresponding point of the conjugate ranges on q by Art. 55. Therefore

$$Q'P.Q'P_2=Q'U.Q'U=Q'U^2.$$

If however q does not cut the circle in real points, Q' is outside the circle. With centre Q' and radius = tangent from Q' to the circle describe another circle. This latter circle will be orthogonal to the given circle. Hence by Art. 52 it meets q at two points, say T, T_2 which are conjugate with regard to the given circle. But we have clearly

$$Q'T=-Q'T_2$$

and
$$Q'P.Q'P_2=Q'T.Q'T_2=-Q'T^2$$
$$=-\text{square of radius of orthogonal circle},$$

that is, P, P_2 are on opposite sides of Q' and $PQ'.Q'P_2=(\text{tangent from } Q' \text{ to given circle})^2$.

59. Self-polar triangles. Let P (Fig. 19) be any point, p its polar with regard to a circle. On p take two conjugate points Q, R. Join $PR=q$, $PQ=r$. Then because Q is on p, the polar of Q passes through P. Also because Q, R are conjugate, the polar of Q passes through R. Therefore the polar of Q is PR,

i.e. q. Similarly the polar of R is r. The triangle PQR is therefore such that each vertex is the pole of the opposite side and its vertices (and likewise its sides) are conjugate in pairs. Such a triangle is said to be *self-conjugate* or *self-polar* with regard to the circle.

Given a self-polar triangle PQR and a point A on the conic, three other points B, C, D can be found. For join PA meeting QR at E. Take B harmonically conjugate to A with regard to P, E. Then since P is pole of QR, B is on the curve. Similarly join AQ meeting RP at F, AR meeting PQ at G. Join RB meeting AQ at C, QB meeting AR at D. $\because$ $Q(RAGD)$ is harmonic, QP is a side of the diagonal triangle of the quadrangle $ABCD$, $\therefore$ P is a diagonal point.

Hence $\{DGAR\} = -1$, $\{CFAQ\} = -1$, i.e. D, C are on the

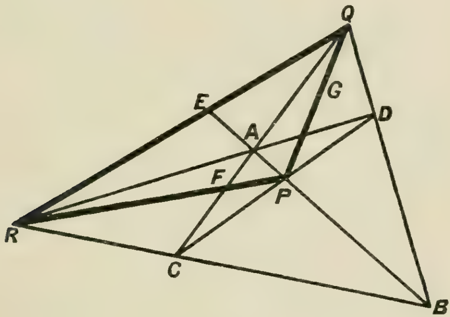


Fig. 19.

conic. And DC passes through P. No new points can therefore be obtained by repeating this construction with B, C or D.

60. Transference of pole and polar properties to the conic. Since properties of incidence and tangency and harmonic properties are not altered by projection, it follows that if we project the circle into any conic all the results which can be stated in terms of properties that are not altered by projection will hold equally of the conic—even though in the proof we may have used properties which do not persist in projection. It is this fact which gives value to the method.

Results which involve the line at infinity, or perpendicular lines, or measurement of lengths (apart from cross-ratio) do not

generally persist in projection and therefore are not necessarily true of the conic.

The following properties follow for the conic.

If a ray through O meet the conic at Q, R the locus of points harmonically conjugate to O with regard to Q, R is a straight line, the polar of O with regard to the conic: O is the pole of the straight line with regard to the conic.

If the polar of P pass through Q the polar of Q passes through P: P, Q are conjugate points with regard to the conic.

If the pole of p lie on q the pole of q lies on p: p, q are conjugate lines with regard to the conic.

Two conjugate lines through a point are harmonically conjugate with regard to the two tangents to the conic from that point.

The polars of a range with regard to a conic form a pencil equi-anharmonic with the range and conversely.

If P, P' be conjugate points on two fixed lines (which may be coincident) the ranges $[P]$, $[P']$ are projective.

If p, p' be conjugate lines through two fixed points (which may be coincident) the pencils $[p]$, $[p']$ are projective.

The polar of a point on the conic is the tangent at that point and the point is conjugate to every other point on the tangent.

Conversely the pole of a tangent is its point of contact and the tangent is conjugate to every other line through this point of contact.

A point and two conjugate points on its polar form a self-polar triangle with regard to the conic.

61. Constructions for pole and polar with regard to a conic. Let O (Fig. 20a) be any point: to construct its polar with regard to a conic draw any two lines through O meeting the conic at P and Q, R and S. Join PR, QS meeting at T; PS, RQ meeting at U. Join TU. Then TU is the polar required.

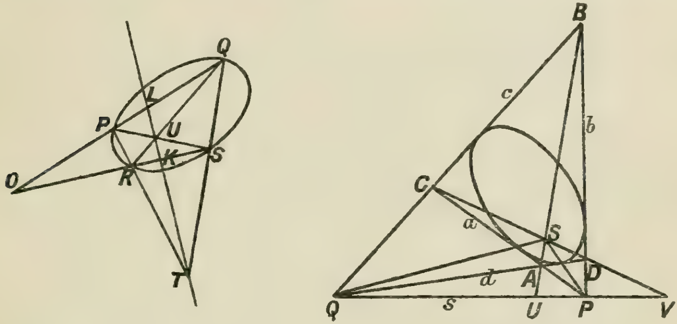
For let TU meet PQ at L, RS at K. By the harmonic property of the quadrangle $PQRS$, the pencil $T(ORKS)$ is harmonic. Hence the ranges $(ORKS)$ $(OPLQ)$ are harmonic, that is K, L are points on the polar of O with regard to the conic. Therefore KL, i.e. TU, is this polar.

Note that the same construction, starting from U, gives OT as the polar of U with regard to the conic and OU as the polar of T. TOU is therefore a self-polar triangle for the conic, or:

The diagonal triangle of any quadrangle inscribed in a conic is self-polar for the conic.

In like manner let s (Fig. 20 b) be any straight line: to construct its pole with regard to a conic.

Take any two points P and Q on s, which lie outside the conic. Draw the tangents a, b to the conic from P and the tangents c, d to the conic from Q. Then S, the intersection of the diagonals (ad, bc) (bd, ac) of the quadrilateral $abcd$, is the pole of s with regard to the conic. For by the harmonic property of the complete quadrilateral, if SUV be the diagonal triangle of the quadrilateral, the diagonal AB is harmonically divided at S and U. Hence $P(UASB)$ is harmonic. Therefore



(a) (b)

Fig. 20.

by Art. 53 the pole of PQ lies on PS. Similarly it lies on QS. Therefore it must be S.

The same construction, starting from SU, taking two points A, B on it, leads to V as the pole of SU, and similarly it leads to U as the pole of SV. SUV is therefore a self-polar triangle with regard to the conic, or:

The diagonal triangle of a complete quadrilateral circumscribed to a conic is self-polar for the conic.

62. Pole and polar with respect to a line-pair and a point-pair. If the conic be a line-pair whose components are a, b, meeting at C, the polar of any point P is the ray p through C harmonically conjugate to CP with regard to a, b. For every line through P meets CP, a, p, b in a harmonic range.

Conversely consider the pole of any line. Let P, Q be two points on that line. The polar of P is a line p through C and

the polar of Q is a line q through C. Hence the pole of PQ is C. Thus every line not through C has C for its pole and every point not C has its polar passing through C.

If the conic be a point-pair whose components are A, B, of which c is the join, the pole of a line p is the fixed point through which pass the harmonic conjugates to p with regard to the two tangents from points of p to the conic. If Q be any point of p, QA, QB are the two tangents from Q to the point-pair. If the harmonic conjugate to p with regard to QA, QB be drawn, it meets c at a fixed point P, which is the harmonic conjugate of pc with regard to A and B. Hence the poles of every line other than c lie on c.

Also the polar of any point not on c is clearly c. For AB is the chord of contact of tangents from every such point.

63. Reciprocal polars. It appears from the previous theory that a conic s establishes a reciprocal correspondence between the elements of its plane, thus: to a point P corresponds its polar p, to a line q corresponds its pole Q. To any figure in the plane, made up of points and lines, will correspond another figure, made up of lines and points, which are the polars and poles respectively of the points and lines of the first figure with regard to the conic s, which is called the *base-conic*.

It will also be seen that properties of incidence are preserved in this reciprocal correspondence, for if P lies on q, p passes through Q; and to the meet of two lines pq corresponds the join of their poles PQ. Also from Art. 54 it follows that cross-ratio properties are unaltered. Accordingly two projective ranges will reciprocate into two projective pencils, self-corresponding points will reciprocate into self-corresponding rays, in particular perspective ranges will reciprocate into perspective pencils, four harmonic points will reciprocate into four harmonic rays; and conversely.

To a curve given as a locus of points will correspond a curve given as an envelope of tangents: the *degree* of one curve —that is, the number of points in which it is met by a straight line—becomes the *class* of the corresponding curve, that is, the number of tangents which can be drawn to it from any point. The join of two coincident points at P on the first curve, i.e. the tangent at P, reciprocates into the meet of two coincident tangents p of the second curve, that is, the point of contact of p.

64. The reciprocal of a conic is a conic. Let the

given conic be obtained as the product of two projective pencils. The two projective pencils reciprocate into two projective ranges and intersections of corresponding rays into joins of corresponding points. The product of two projective pencils therefore reciprocates into the product of two projective ranges, that is, into a conic.

65. Principle of Duality. It follows from the transformation by reciprocal polars that to every theorem concerning a figure made up of points and lines there corresponds another theorem concerning a corresponding figure made up of lines and points respectively, so that geometrical theorems appear in pairs. Several instances of this principle of duality have already been met with and it will be an instructive exercise for the student to trace such dual theorems as have already been given. As examples of such theorems we may quote the following, corresponding theorems appearing side by side:

If in two corresponding figures meets of corresponding lines lie on a fixed line, joins of corresponding points pass through a fixed point.	If in two corresponding figures joins of corresponding points pass through a fixed point, meets of corresponding lines lie on a fixed line.
The harmonic property of the complete quadrangle.	The harmonic property of the complete quadrilateral.
The meets of cross-joins of corresponding points of two projective ranges lie on a fixed line.	The joins of cross-meets of corresponding rays of two projective pencils pass through a fixed point.
The harmonic conjugates of a fixed point with regard to the two points at which any line through it meets a fixed conic lie on a fixed line.	The harmonic conjugates of a fixed line with regard to the two lines which can be drawn from any point on it to touch a fixed conic pass through a fixed point.

Reciprocal theorems are obtained at once one from the other by simply translating the language, the following being the terms interchanged:

straight line	point
join	meet
tangent to a curve	point on a curve
point of contact of a tangent	tangent at a point on the curve
lie on	pass through
range	pencil
collinear	concurrent
degree	class
locus	envelope

It should be noticed, however, that theorems true of *special*

curves reciprocate into theorems true only of the curves which are the reciprocals of these special curves. Also that properties of length and angular magnitude (which are termed metrical properties) do not generally reciprocate into like properties. It will be found that the properties to which the principle of duality can be applied successfully are the *projective* properties.

66. Centre and diameters of a conic. The pole of the line at infinity with regard to a conic is called the *centre* of the conic. The centre of the conic corresponds, in the plane of the original circle, not to the centre of the circle, but to the pole of the vanishing line.

Lines through the centre of a conic are called its *diameters*. Since the centre and the point at infinity divide a diameter harmonically, a diameter is bisected at the centre.

Conjugate diameters are conjugate lines through the centre. Hence the pole of either is the point at infinity on the other. Therefore the tangents at the extremities of a diameter are parallel to its conjugate.

By the pole and polar property chords parallel to a diameter are divided harmonically by the point at infinity on the chords and by the conjugate diameter, that is, they are bisected by the conjugate diameter.

If C be the centre of a conic, P any point, the polar p of P with regard to the conic is conjugate to the diameter CP and therefore bisected by it. For if c be the line at infinity, clearly pc, that is, the point at infinity on p, is the pole of PC.

All diameters of an ellipse meet the curve in real points. For the vanishing line being outside the circle, its pole is inside. Every line through this pole therefore cuts the circle in real points, and the same holds good after projection.

On the other hand, of two conjugate diameters of a hyperbola, one and one only meets the curve in real points. For consider the original circle. The vanishing line cuts the circle in real points I, J (Fig. $21a$). The tangents at I, J meet at C, which is the pole of the vanishing line and is outside the circle. Of the rays through C, those which lie inside the angle ICJ meet the circle, the others do not. Now by Art. 53 any two conjugate lines through C are harmonically conjugate with regard to CI, CJ. If they meet IJ at P, P', $(PIP'J)$ is harmonic. Hence if P' be inside IJ, P is outside and conversely, since P, P' divide IJ internally and externally in the same ratio (Art. 28); $\therefore$ if CP' cuts the circle in real points, CP does not, and conversely.

Projecting IJ to infinity the property stated follows for diameters of a hyperbola.

Also note that in the hyperbola the property that CP, CP' are harmonically conjugate with regard to the tangents from C becomes:

Two conjugate diameters of a hyperbola are harmonically conjugate with regard to the asymptotes.

In the case of the parabola the vanishing line c touches the original circle at C (Fig. 21 b), and every line CP is conjugate to c. The centre of the parabola is therefore at infinity, and its direction gives the point of contact of the line at infinity with the curve. All diameters of a parabola are parallel to this fixed

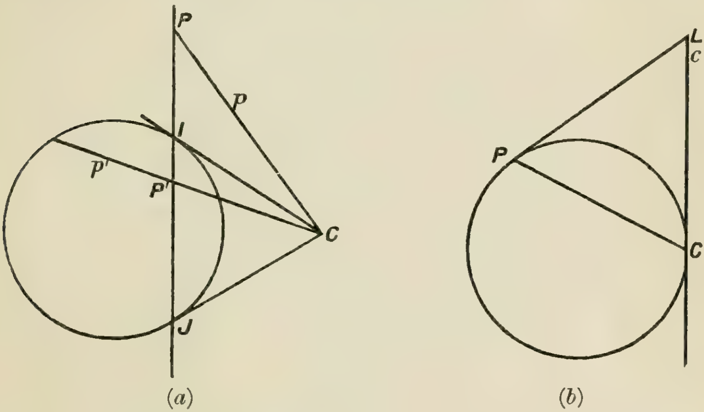


Fig. 21.

direction, and are to be looked upon as conjugate to the line at infinity. The line at infinity has no definite direction, but it may be shown that to each diameter there is a definite conjugate *direction*. For let L (Fig. 21 b) be the pole of CP, chords through L are conjugate to CP. Project c to infinity: the circle becomes a parabola, CP a diameter, the chords through L a system of parallel chords bisected by that diameter, PL the tangent at its extremity, which tangent is parallel to the chords. Hence a diameter of a parabola bisects chords parallel to the tangent at its extremity.

Because the ellipse and hyperbola have their centre at a finite distance, they are termed *central* conics.

67. Supplemental chords of a central conic are parallel to conjugate diameters. Let ACB (Fig. 22) be a diameter of a conic. If P be any point on the conic, PA, PB are known as supplemental chords. If D be the middle point of PB, then, C being the middle point of AB, CD is parallel to PA. Also it is the diameter conjugate to the direction PB since it bisects PB. Therefore PA, PB are parallel to conjugate diameters. Conversely, if through A, B parallels be drawn to two conjugate directions, they meet at a point of the curve. For if CD be the diameter conjugate to BP meeting BP at D, the point P on BP whose distance from $D = DB$

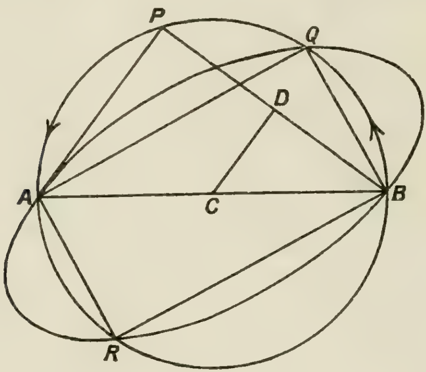


Fig. 22.

must lie on the curve (since CD bisects the chord), hence the parallel through A to CD meets BP at P on the curve.

68. Axes of a conic. An *axis* of a conic is a diameter perpendicular to its conjugate.

To prove that a central conic has one and only one pair of axes.

Let ACB (Fig. 22) be a diameter of the conic. Assuming for the moment the existence of a pair of axes, draw through A a parallel to one of them, through B a parallel to the other. The intersection Q of these two lines lies on the conic, by the preceding article. It also lies on a circle on AB as diameter. Hence the intersections of supplemental chords parallel to a pair of axes are the intersections of the conic with the circle on AB as diameter.

The circle and conic have already two intersections, viz. at A and B. Hence, by Art. 45, they can have at most two other intersections. Now a circle has an inside and an outside, and to pass from inside to outside it is necessary to cross the circle. Projecting on to any plane and bearing in mind that projection does not alter the relative *order* of points, we see that a conic has also an inside and an outside, which are the projections of the inside and outside, respectively, of the original circle. And to pass from inside to outside, or conversely, it is necessary to cross the conic.

Now let us go round the circle of Fig. 22 in an assigned sense, indicated by the arrows in the figure. If the tangents to the conic at A and B (which are parallel, being tangents at the extremities of a diameter) are not perpendicular to AB, the circle crosses the conic at A and B. Owing to the central symmetry of both conic and circle, a rotation of two right angles about C brings both figures into coincidence with themselves, the points A and B being interchanged. Hence, if at A the circle crosses the conic from outside to inside, it crosses at B also from outside to inside. Therefore between A and B it crosses from inside to outside, giving an intersection R, and between B and A it crosses from inside to outside, giving an intersection Q. A, B, Q, R are then the four real intersections of the circle and conic.

Also the point symmetrical to Q with respect to C lies on both circle and conic. It must accordingly be R. QR, AB bisect each other at C, $\therefore$ $AQBR$ is a parallelogram and, since the angle at Q is a right angle, it must be a rectangle.

Hence the points Q, R give the same pair of axes, namely the parallels to the sides of $AQBR$.

Suppose now the tangents at A and B to the conic are perpendicular to AB. Then the circle and conic touch at A and B, that is, they have four real common points, coincident in pairs at A and B. They have then no other intersections. AB is an axis, being perpendicular to the tangent at A, and no axes exist, other than AB and the perpendicular to AB.

Thus for any central conic there exists one pair of axes, which are always real, and one pair only.

Both axes of an ellipse meet the curve in real points; the longer and shorter axes are called the *major* and *minor* axes of the ellipse respectively.

By Art. 66, one axis of a hyperbola meets the curve in real points. This is called the *transverse* axis. The axis which does not meet the curve in real points is called the *conjugate* axis.

In the parabola an axis is a diameter which bisects chords perpendicular to itself. Since all diameters are parallel, we have to take that one which bisects chords perpendicular to all diameters. Hence a parabola has only one axis.

The points where an axis meets a conic are called *vertices* of the curve.

Note that a conic is *symmetrical* with regard to each axis.

For if P be a point on the curve the chord through P perpendicular to an axis is bisected by that axis and therefore meets the curve again at the symmetrical point P'.

Also the axes are harmonically conjugate with regard to the asymptotes, and therefore (being perpendicular) they bisect the angles between the asymptotes by Art. 28.

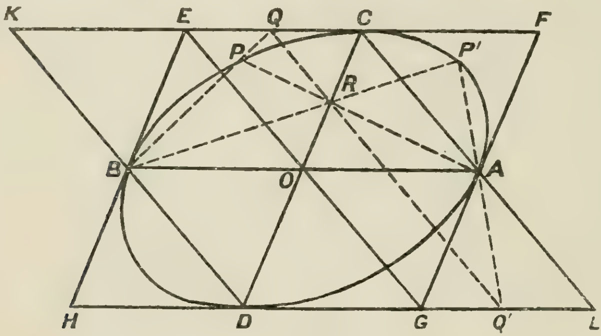


Fig. 23.

69. Construction of an ellipse when two pairs of conjugate diameters are given in position and length. Let AOB, COD be the two conjugate diameters (Fig. 23). Complete the parallelogram $EFGH$, of which they are median lines. Draw any line QRQ' parallel to the diagonal EG of this parallelogram meeting EF, CD, GH at Q, R, Q' respectively. Join AR, BQ meeting at P. $\because$ QR is parallel to a fixed line, the ranges $[Q]$, $[R]$ are similar and therefore projective. Hence $A[R] \barwedge B[Q]$. The locus of P is therefore a conic. This conic passes through A and B, these being vertices of the two pencils. Also if QR is along EG, Q is at E, R is at O, hence AB corresponds to BE: if QR be at infinity, Q is at infinity on FE, i.e. on BA, R is at infinity on DC, i.e. on AF; hence BA corresponds to AF. The conic locus of P $\therefore$ touches EH at B,

FG at A. And it passes through C, as is obvious by taking QR through C, when Q, R, P coincide at C. It has therefore five points common with the required ellipse, viz. two coincident points at A, two at B and one at C. Hence it is the required ellipse.

By taking lines QR between CA and BD the half of the ellipse inside $CEHD$ can be drawn in this way. To avoid taking distant parallels QR and to keep the construction compact, the other half of the ellipse may be drawn by joining BR, AQ' meeting at P'. P' can be shown to be a point on the ellipse by reasoning similar to that used above, and the same set of parallels can be employed to complete the ellipse.

EXAMPLES IVa.

1. Without using the polar property, prove directly that if p be any fixed line in the plane of a circle, P any point on p outside the circle, t, t' the two tangents from P to the circle, the line p' harmonically conjugate to p with regard to t, t' passes through a fixed point.

2. Show that the cross-centre of two projective pencils is the pole with regard to their product of the join of their vertices.

3. Show that the cross-axis of two projective ranges is the polar with regard to their product of the meet of their bases.

4. Given two points A, B on a conic and the pole C of AB, construct the tangent to the conic at any given point P.

5. Given two tangents a, b to a conic and the polar c of ab, construct the point of contact of any other tangent p.

6. Two ranges of conjugate points with regard to a conic lie on straight lines s, s'. Show that the cross-axis of the ranges passes through the poles S, S' of the lines.

7. ABC is any triangle. A' is the pole of BC with regard to a conic, B' is the pole of CA, C' is the pole of AB. Show that the triangles ABC, $A'B'C'$ are in plane perspective.

[Use Ex. 6 noting that B, C' and B', C are pairs of conjugate points.]

8. Of the three sides of a triangle self-polar with regard to a conic, two meet the curve in real points and one does not.

9. Given a point and its polar with regard to a conic and a point on the curve, find another point on the curve.

10. If P and Q be points on the radical axis of two circles and P, Q be conjugate points for one circle, they are also conjugate for the other.

[The radical axis of two circles is the locus of points the tangents from which to the two circles are equal. It is the common chord if the circles cut in real points.]

11. If through two points A and B conjugate pencils be drawn with regard to a conic, the product of these conjugate pencils is a conic passing through the points of contact of the tangents from A, B to the original conics.

12. If on two lines a, b conjugate ranges be taken with regard to a conic, the product of these conjugate ranges is a conic touching the tangents to the original conic at the points where the latter is met by a, b.

13. A, B are two fixed points in the plane of a conic s. P is a point such that the two tangents from P to s are harmonically conjugate with regard to PA, PB. Show that the locus of P is a conic.

14. State and prove the theorem obtained from Ex. 13 by reciprocation.

15. The locus of the intersections of the polars with regard to two fixed circles of a point P lying on a fixed straight line is a conic.

16. The product of two conjugate pencils with regard to a circle, the vertex of one of which is the centre, is a circle.

17. If T be any point, C the centre of a conic, N the point where the polar of T meets the diameter through T, A a point where this diameter meets the conic, prove that $CN \cdot CT = CA^2$.

18. If a diameter of a parabola meet the curve at P and a conjugate chord at V, show that if T be the pole of that chord, T lies on the diameter and $TP = PV$.

19. If two pairs of conjugate diameters of a conic are at right angles the conic is a circle.

20. Show that a conic is completely determined if two points and their polars and a point on the curve be given.

21. Show that a conic is completely determined if two points and their polars and a tangent to the curve be given.

22. Prove the following construction for a conic, given a diameter AB, a point P on the curve and the direction conjugate to AB. Complete the parallelogram $ADPE$ on AP as diagonal and whose sides AD, DP are along and conjugate to AB respectively. Let a parallel to DE meet PD at Q, PE at R. The rays AR, BQ meet on the conic.

23. Given a self-polar triangle for a conic and a tangent to the conic, construct three other tangents.

24. A pair of conjugate diameters of a given conic meet a given straight line at A and B; on AB is described a triangle APB similar to a given triangle. Prove that the locus of P is a hyperbola and find its asymptotes.

25. Through a fixed point O a straight line is drawn to meet a fixed straight line l at P and intersects the polar of P with respect to a fixed conic s at Q. Show that as P describes the straight line l, Q describes a conic passing through three fixed points independent of the line l chosen.

Show that (1) inversion is a particular case of this construction and that (2) when O lies on s the conics corresponding to two lines such as l of the plane have simple contact with s at O, but contact of the second order with each other.

[P, P' are said to be *inverse* points with regard to O if (O, P, P' being collinear) $OP \,.\, OP' = \text{const.}$]

EXAMPLES IV B.

1. Two conjugate diameters of an ellipse are respectively 8″ and 6·4″ in length and the angle between them is 110°; draw the ellipse, and measure the lengths of its principal axes.

2. Using the ruler only construct the polars of the points (1, 1) and (6, 2) with regard to the circle $x^2+y^2=9$. In any manner construct the polars of the point (3, 0) and of the points at infinity on $x=0$ and $y=x$ with regard to the same circle.

3. Construct a line passing through the point (3, 0) and conjugate to $y=0$ with regard to the circle $(x-3)^2+(y+2)^2=1$.

4. Construct the envelope of the polar of a point P on the circle

$$x^2-4x+y^2=0$$

with regard to the circle $x^2+y^2=4$.

5. Draw the conic through the five points (0, 3) (0, 5) (1, 0) (4, 0) (2, 2) and construct its axes.

6. If the pole of perspective be (1, −2), the axis of collineation $x=-2$, construct the axis of the parabola in plane perspective with the circle $x^2+y^2-3x=0$, the vanishing line for the circle being $x=0$.

7. The coordinates being rectangular, A, B are the points (−4, 0), (3, 0) respectively. If AR be any ray through A, P the pole of AR with regard to the circle $x^2+y^2=4$, and if BP meet AR at Q, construct the locus of Q.

CHAPTER V.

NON-FOCAL PROPERTIES OF THE CONIC.

70. Pascal's Theorem. If $AB'CA'BC'$ (Fig. 24) be a hexagon* inscribed in a conic, the meets of opposite sides

$$(AB', A'B), (BC', B'C), (CA', C'A)$$

are collinear.

Let P be $(AB', A'B)$; $Q=(BC', B'C)$; $R=(CA', C'A)$; $L=(AC', BA')$; $M=(BC', CA')$.

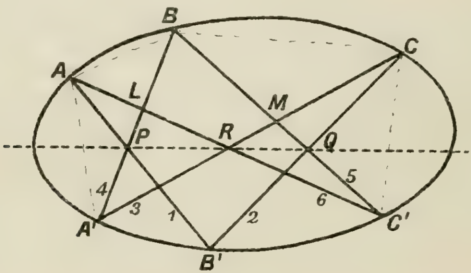


Fig. 24.

Project the four points A', B', C', B from A, C: we have by the property of the conic

$$A(A'B'C'B) \barwedge C(A'B'C'B).$$

Cutting the first pencil by $A'B$, the second by BC',

$$(A'PLB) \barwedge (MQC'B).$$

* The hexagon considered here is not generally, and in graphical examples not conveniently, a *convex* figure. A similar remark applies to polygons in general, except where the contrary is distinctly stated.

These two projective ranges have a self-corresponding point B, $\therefore$ they are perspective and the joins of corresponding points are concurrent,

$\therefore$ $A'M$, PQ, $C'L$ are concurrent,

$\therefore$ $(A'M, C'L)$ lies on PQ, $\therefore$ R lies on PQ.

71. Brianchon's Theorem. If $ab'ca'bc'$ be a hexagon circumscribed to a conic, the joins of opposite vertices

$$p\,(=(ab', a'b)),\;\; q\,(=(bc', b'c)),\;\; r\,(=(ca', c'a))$$

are concurrent.

This theorem is obtained immediately from Pascal's theorem by reciprocation. The student will find it instructive to construct a proof of Brianchon's theorem from the proof given above of Pascal's theorem, reciprocating each step.

Pascal's and Brianchon's theorems are conveniently expressed by the following numerical rule:

Pascal. If 1, 2, 3, 4, 5, 6 be the sides of a hexagon inscribed in a conic taken in order, then 14, 25, 36 are collinear.

The line on which they lie is called the *Pascal line* of the inscribed hexagon.

Brianchon. If 1, 2, 3, 4, 5, 6 be the vertices of a hexagon circumscribed to a conic taken in order, then 14, 25, 36 are concurrent.

The point through which they pass is called the *Brianchon point* of the circumscribed hexagon.

72. Construction of conic through five points. By means of Pascal's theorem we can construct the conic through five points.

Take the points in any convenient order, letter them in this order $AB'CA'B$. Number the *sides* $AB' = 1$, $B'C = 2$, $CA' = 3$, $A'B = 4$. Then $P = 14$ in Pascal's theorem is known. Draw any Pascal line PRQ meeting 2 at Q and 3 at R. Join Q to the free end of 4, viz. B, and R to the free end of 1, viz. A. The intersection of AR, BQ is a point C' on the conic.

By taking various Pascal lines through P we can construct any number of points on the conic.

73. Construction of conic touching five lines. Similarly let five tangents to a conic be given. Letter them in order $ab'ca'b$. Number the *vertices* $ab' = 1$, $b'c = 2$, $ca' = 3$, $a'b = 4$. Then $p = 14$ is a fixed line. On p take any Brianchon point B. Let q be the join of B to 2, r the join of B to 3.

q meets b the open side through 4 at the vertex 5, r meets a the open side through 1 at the vertex 6. 56 is the tangent c' to the conic.

By taking different Brianchon points on p, we can construct the conic by tangents as an envelope.

74. Coincident elements. Important particular cases of Pascal's and Brianchon's theorems occur when two elements coincide. In this case it is important to bear in mind that if the coincident elements are points, these points have to be taken as consecutive vertices of the Pascal hexagon and the side of the hexagon joining them is to be interpreted as the corresponding tangent. If the coincident elements are tangents, these are consecutive sides of a Brianchon hexagon, and the vertex of the hexagon common to them is interpreted as the corresponding point of contact.

In all cases we shall write repeated elements twice over when considering Pascal and Brianchon hexagons, thus

$$AABCDD$$

will be considered a hexagon, and its sides taken in order are

AA (tangent at A),
AB,
BC,
CD,
DD (tangent at D),
DA.

75. Asymptote properties of the Hyperbola. Let C (Fig. 25) be the centre of a hyperbola, A^∞, B^∞ the points at infinity on the two asymptotes, P, Q two points on the curve. Consider the Pascal hexagon $A^\infty A^\infty PB^\infty B^\infty Q$; its sides taken in order are as follows: $A^\infty A^\infty = 1 =$ the asymptote CA; $A^\infty P = 2 =$ the parallel PL to CA; $PB^\infty = 3 =$ the parallel PM to CB; $B^\infty B^\infty = 4 =$ asymptote CB; $B^\infty Q = 5 =$ the parallel QL to CB; $QA^\infty = 6 =$ the parallel QM to CA.

Hence $14 = C$, $25 = L$, $36 = M$ and C, L, M are collinear, that is, if on PQ as diagonal a parallelogram be described whose sides are parallel to the asymptotes the other diagonal passes through the centre.

It follows that the parallelograms $PNCN'$, $QKCK'$, being

made up of $LK'CN$ and of the complements $LPN'K'$, $KQLN$ respectively, are equal in area. Hence if through a point P on a hyperbola parallels are drawn to the asymptotes, this parallelogram is of constant area.

Also $K'N$ is parallel to PQ, for the parallelograms $CNLK'$, $LQMP$ are clearly similar and similarly placed. Hence if PQ meet the asymptotes at R, S, $PR = NK' = QS$ (opposite sides of parallelograms). Hence the distances intercepted on any straight line between the curve and the asymptotes are equal.

This last property furnishes an easy method for drawing a

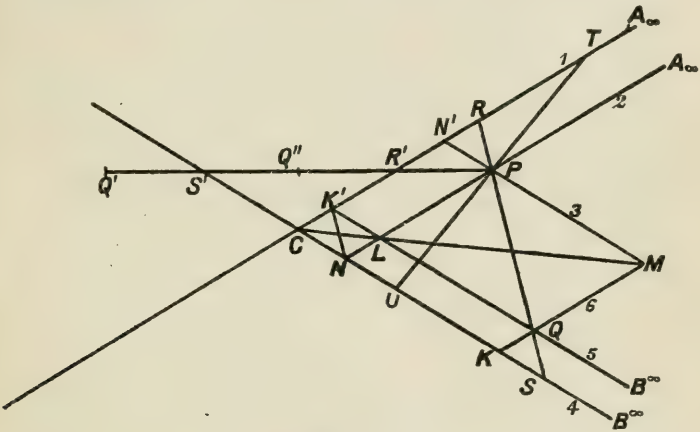


Fig. 25.

hyperbola when the asymptotes and one point P on the curve are given. Draw a variable ray through P meeting the asymptotes at R and S. On this ray take a point Q such that $SQ = PR$. Q describes the hyperbola.

Care must be taken that in all cases PQ and RS shall have the *same mid-point*. Thus in Fig. 25, when the ray is drawn as $PR'S'$, Q must be taken at Q' outside $S'R'$ and *not* at Q'' inside.

If the points P, Q coincide the property last proved becomes: the intercept of a tangent to a hyperbola between the asymptotes is bisected at the point of contact.

If TU (Fig. 25) be drawn through P parallel to NN', $TP = NN' = PU$. Hence TU is the tangent at P.

Also the triangle TCU = twice parallelogram $PNCN'$
= const. by property proved above.

Hence a variable tangent to a hyperbola cuts off from the asymptotes a triangle of constant area.

76. Construction of a hyperbola, given three points and the directions of both asymptotes. We first of all proceed to construct the centre.

If A, B, C be the three given points, construct the parallelograms on AB, BC as diagonals whose sides are parallel to the asymptotes. The centre is then the intersection of the other two diagonals (Art. 75). The asymptotes are now known in position and the hyperbola may be constructed by the method of Art. 75.

77. Given four points on a hyperbola and the direction of one asymptote, to construct the direction of the other asymptote. Let A, B, C, D be the four points; let E^∞ be the direction of the given asymptote, F^∞ that of the required asymptote. Then, considering the hexagon $ABCDE^\infty F^\infty$, the points P = intersection of AB, DE^∞, Q^∞ = intersection of BC and line at infinity, R = intersection of CD, $F^\infty A$ are collinear. Hence if the parallel through P to BC meet CD at R, AR gives the direction required. We can now use the method of Art. 75 to construct the asymptotes in position and hence to draw the hyperbola.

78. Parabola from four tangents. Since the line at infinity i^∞ is a tangent to the parabola, four tangents a, b, c, d define the curve. Let t be any required tangent. Consider the Brianchon hexagon $i^\infty abcdt$ (Fig. 26). Let 1, 2, 3, 4, 5, 6 be the vertices $i^\infty a$, ab, bc, cd, dt, ti^∞ in order. p or 14 is then the parallel through cd to a. On this take any Brianchon point B. Join $2B$ meeting d at 5: the parallel through 5 to $3B$ is the tangent required.

By this method we can draw a tangent to a parabola parallel to any required direction. For draw through 3 a parallel r to this direction to meet p at B; B is the corresponding Brianchon point.

Also we can construct at once the direction of the axis. For we have to find the point of contact of the line at infinity. To do this consider the Brianchon hexagon $abcdi^\infty i^\infty$. We have (ab, di^∞) $(bc, i^\infty i^\infty)$ $(cd, i^\infty a)$ are concurrent. Hence if the parallels through cd to a and through ab to d meet at E and

bc is F, EF goes through the point of contact of i , that is, it is parallel to the axis.

The tangent perpendicular to the axis is then constructed.

The point of contact of a tangent t is readily found when we know two other tangents and the direction of the axis. For consider the hexagon $ttai^\infty i^\infty b$; $(tt, i^\infty i^\infty)$ (ta, bi^∞) (ai^∞, bt) are concurrent. Hence through the meets of the tangent t with each of the given tangents draw a parallel to the other tangent. The line drawn parallel to the axis through the intersection of these parallels meets the tangent t at its point of contact.

Construct therefore the point of contact of the tangent perpendicular to the axis. This is the vertex of the parabola.

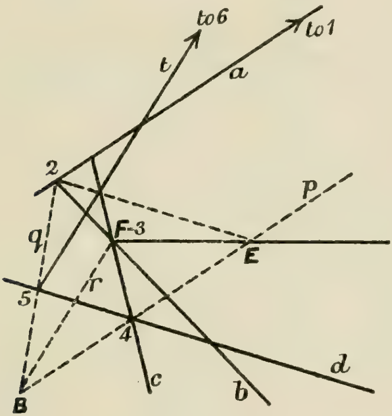


Fig. 26.

The line through the vertex in the direction of the axis is the axis.

79. Parabola from three points and direction of axis. Let three points A, B, C be given and the point I^∞ on the axis, i.e. the direction of the axis. We first construct a point D on the line through A perpendicular to the axis (Fig. 27). Considering the Pascal hexagon $ABI^\infty I^\infty CD$ we have $(AB, I^\infty C)$, (BI^∞, CD), $(I^\infty I^\infty, DA)$ are concurrent. But $I^\infty I^\infty$ is the tangent at I^∞ to the parabola and is therefore the line at infinity. Thus if P is the meet of AB and the parallel to the axis through C, Q the meet of CD and the parallel to the

axis through B, R^∞ the point at infinity on DA, then P, Q, R^∞ are collinear or PQ is parallel to DA. Now P is fixed, A, B, C being given; PQ being perpendicular to the axis, Q is found and CQ meets the perpendicular to the axis through A at the point D required. The line bisecting AD at right angles is therefore the axis of the parabola.

Let V be the vertex. Consider the Pascal hexagon

$$ABI^\infty I^\infty VC.$$

Then $$(AB, VI^\infty), (BI^\infty, VC), (I^\infty I^\infty, CA)$$

are collinear. Let AB meet the axis at E, CA meet the line at

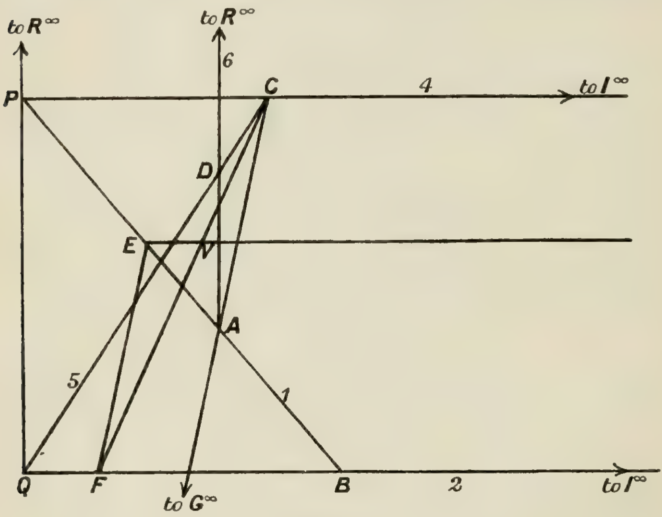


Fig. 27.

infinity at G^∞, VC meet the parallel to the axis through B at F. Then F lies on the parallel to CA through E and V lies on FC. Thus V is known.

80. Parabola from a tangent and its point of contact, another point and the direction of the axis. Let a, A represent the tangent and its point of contact (Fig. 28), B the other point, I^∞ the point at infinity on the axis, M any other point on the curve.

Two constructions may be used, according as we prefer to describe the curve by rays through A or by rays through B.

In the first construction draw lines QR parallel to the tangent at A to meet the parallel to the axis through B and AB at Q, R respectively. Then AQ meets the parallel to the axis through R at a point M on the curve. The result follows by considering the hexagon $AAMI^\infty I^\infty B$; 14 is P^∞, the point at infinity on the tangent at A : $P^\infty QR$ is then the Pascal line.

In the second construction draw a parallel to AB to meet AB at P'^∞, the tangent at A at Q' and the parallel to the axis

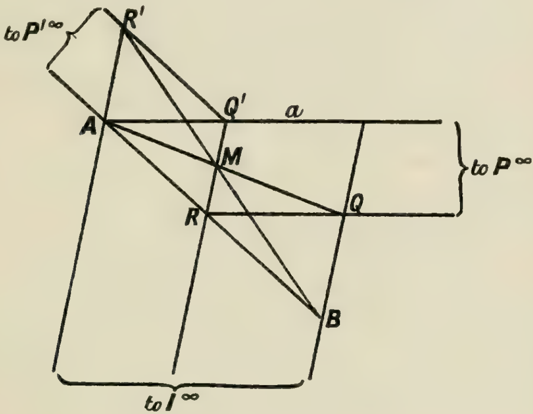


Fig. 28.

through A at R'. Join BR' meeting the parallel to the axis through Q' at a point M. Then M is on the parabola. For $P'^\infty Q'R'$ is the Pascal line of the hexagon $BAAI^\infty I^\infty M$.

81. Inscribed Quadrangle and circumscribed Quadrilateral. If $ABCD$ be a quadrangle inscribed in a conic, consider the Pascal hexagon $AABCCD$. Then

$$(AA, CC), (AB, CD), (BC, AD)$$

are collinear. Hence the tangents at any two given vertices of a quadrangle inscribed in a conic meet on that side of its diagonal triangle which is opposite to the diagonal point lying on the join of the two given vertices.

In like manner, from Brianchon's Theorem, if a, b, c, d be four tangents, considering the hexagon *aabccd*, then

$$(aa, cc), (ab, cd), (bc, ad)$$

are concurrent. Hence the join of the points of contact of any two given sides of a quadrilateral circumscribed about a conic passes through that vertex of its diagonal triangle which is opposite to the diagonal through the meet of the two given sides.

If a, b, c, d be the tangents at A, B, C, D, (ac), (bd) both lie on the join of (AB, CD), (BC, AD). Hence the diagonal points of the complete quadrangle formed by the four points lie on the diagonals of the complete quadrilateral formed by the tangents at these points, or:

The complete quadrilateral formed by four tangents to a conic and the complete quadrangle formed by their four points of contact have the same diagonal triangle.

82. Inscribed and circumscribed triangles. Let ABC be a triangle inscribed in a conic.

From the Pascal hexagon $AABBCC$ we find

$$(AA, BC), (AB, CC), (BB, CA)$$

are collinear, or the sides of an inscribed triangle meet the tangents at the opposite vertices at collinear points.

If *abc* be a triangle circumscribed about a conic, it follows in like manner from the Brianchon hexagon *aabbcc* that the joins of the vertices to the points of contact of the opposite sides are concurrent.

83. Newton's Theorem on the product of segments of chords of a conic. If POP', QOQ' (Fig. 29) be any two chords of a conic intersecting at O, whose directions are given, the ratio of the rectangles $\dfrac{OQ \,.\, OQ'}{OP \,.\, OP'}$ is constant. For suppose the chord QOQ' to move parallel to itself, while the chord POP' remains fixed. Let the line through P parallel to QOQ' meet the conic again at T. Then since the diameter conjugate to QOQ' bisects QQ' and PT and these are parallel, it is obvious from Elementary Geometry that PQ', TQ meet at S on this diameter. The pencils $P\,[Q']$, $T\,[Q]$ are then perspective and therefore projective. Hence by the property of the conic (Art. 37) the pencils $P\,[Q]$, $P'\,[Q']$ are projective. These

pencils determine on any parallel u to QQ' two projective ranges $[R]$, $[R']$. Let u meet PP' at U and let V^∞ be the point at

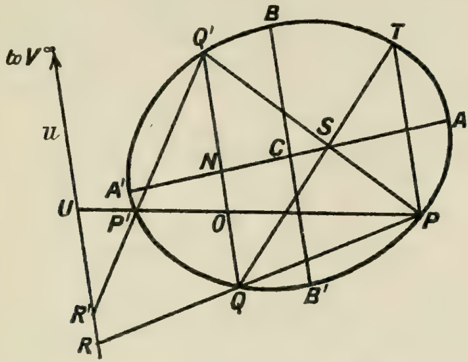


Fig. 29.

infinity on u. Then when Q is at P', R is at U and R' is at V^∞; and when Q' is at P, R is at V^∞ and R' is at U.

Hence if R_1, R_1'; R_2, R_2' be pairs of corresponding points of the ranges $[R]$, $[R']$ we have the set of four points $UR_1V^\infty R_2$ corresponding to the set of four $V^\infty R_1'UR_2'$. Equating the cross-ratios

$$\frac{UR_1 \,.\, V^\infty R_2}{UR_2 \,.\, V^\infty R_1} = \frac{V^\infty R_1' \,.\, UR_2'}{V^\infty R_2' \,.\, UR_1'},$$

or

$$\frac{UR_1}{UR_2} = \frac{UR_2'}{UR_1'};$$

that is

$$UR_1 \,.\, UR_1' = UR_2 \,.\, UR_2'.$$

$$\therefore\ UR \,.\, UR' = \text{const.}$$

But, from Fig. 29, OQ, UR being parallel and OQ', UR' being parallel

$$\frac{OQ}{OP} = \frac{UR}{UP} \text{ and } \frac{OQ'}{OP'} = \frac{UR'}{UP'}.$$

Hence

$$\frac{OQ \,.\, OQ'}{OP \,.\, OP'} = \frac{UR \,.\, UR'}{UP \,.\, UP'} = \text{const.}$$

The ratio $\dfrac{OQ\,.\,OQ'}{OP\,.\,OP'}$ is therefore not altered by shifting the chord QOQ' parallel to itself. Similarly it is not altered by shifting the chord POP' parallel to itself. It is therefore independent of the position of O, once the directions of the chords PP', QQ' are fixed.

If we take O at the centre C the parallel chords are bisected at the centre.

Hence $\dfrac{OQ\,.\,OQ'}{OP\,.\,OP'}$ = ratio of the squares of the parallel semi-diameters.

If $Q' = Q$, $P' = P$ we have: two tangents to a conic from any point are in the ratio of the parallel semi-diameters.

84. Oblique ordinate and abscissa referred to conjugate diameters. If the chord PP' (Fig. 29) coincide with the diameter AA' conjugate to the chord QQ' the property of the last article takes the form

$$\frac{NQ\,.\,NQ'}{NA\,.\,NA'} = \text{const.} = \frac{CB^2}{CA^2},$$

CB being the semi-diameter conjugate to CA; or since

$$NQ' = -NQ,$$

$$\frac{QN^2}{AN\,.\,NA'} = \text{const.} = \frac{CB^2}{CA^2}.$$

In Fig. 29 the conic is an ellipse, and the diameter BCB' meets the curve in real points. Therefore CB^2 is positive and $AN\,.\,NA'$ is positive, so that N lies between A and A'.

But if the conic be a hyperbola we know that if ACA' meet the curve in real points, its conjugate BCB' does not meet the curve. Hence there is no real semi-diameter CB.

Nevertheless the theorem of Art. 83 holds good and

$$\frac{QN^2}{AN\,.\,NA'} = \text{a constant},$$

but N is outside AA' and the constant is negative. If we then construct a length CB_1 such that

$$-\frac{CB_1^{\,2}}{CA^2} = \frac{QN^2}{AN\,.\,NA'},$$

and lay it off along the diameter conjugate to AA', CB_1 may be

spoken of as the real length, or simply the length, of the semi-diameter conjugate to CA. But it should be carefully remembered that B_1 is *not* a point on the hyperbola.

Returning to the relation

$$\frac{QN^2}{AN \, . \, NA'} = \frac{CB^2}{CA^2}$$

for the ellipse, we have

$$\frac{QN^2}{CB^2} = \frac{(AC + CN)(NC + CA')}{CA^2}$$

$$= \frac{-(CN - CA)(CA + CN)}{CA^2}$$

$$= \frac{CA^2 - CN^2}{CA^2},$$

or $$\frac{CN^2}{CA^2} + \frac{QN^2}{CB^2} = 1.$$

For the hyperbola

$$\frac{QN^2}{CB_1^2} = -\frac{CA^2 - CN^2}{CA^2},$$

or $$\frac{CN^2}{CA^2} - \frac{QN^2}{CB_1^2} = 1.$$

In the case of the parabola A' is at infinity.

Take two chords Q_1Q_1', Q_2Q_2' conjugate to the same diameter,

$$\frac{Q_1N_1^2}{AN_1 \, . \, N_1A'} = \frac{Q_2N_2^2}{AN_2 \, . \, N_2A'},$$

$$\frac{Q_1N_1^2}{AN_1} = \frac{Q_2N_2^2}{AN_2} \cdot \frac{N_1A'}{N_2A'};$$

now let A' be taken at infinity $N_1A' : N_2A' = 1$.

Hence $$\frac{Q_1N_1^2}{AN_1} = \frac{Q_2N_2^2}{AN_2},$$

or $$\frac{QN^2}{AN} = \text{constant for the parabola.}$$

The above relations lead to the well-known analytical equations of the ellipse, hyperbola and parabola referred to conjugate directions.

85. Carnot's Theorem. If the sides BC, CA, AB of a triangle ABC meet a conic at P, P'; Q, Q'; R, R' respectively, then

$$\frac{BP \cdot BP'}{CP \cdot CP'} \cdot \frac{CQ \cdot CQ'}{AQ \cdot AQ'} \cdot \frac{AR \cdot AR'}{BR \cdot BR'} = 1.$$

For let OL, OM, ON be the semi-diameters of the conic parallel to BC, CA, AB respectively. Then by Art. 81

$$\frac{AR \cdot AR'}{AQ \cdot AQ'} = \frac{ON^2}{OM^2}; \quad \frac{BP \cdot BP'}{BR \cdot BR'} = \frac{OL^2}{ON^2}; \quad \frac{CQ \cdot CQ'}{CP \cdot CP'} = \frac{OM^2}{OL^2}.$$

Multiplying these three relations together the result follows.

86. Intersections of a conic and a circle. Let a circle meet a conic at four points P, Q, R, S, and let PQ meet RS at O.

Then if CL, CM are the semi-diameters of the conic parallel to PQ, RS we have

$$\frac{CL^2}{CM^2} = \frac{OP \cdot OQ}{OR \cdot OS} = 1$$

by the property of segments of chords of a circle.

Hence the semi-diameters CL, CM are equal. Now the extremities of all equal semi-diameters lie on a circle concentric with the conic. Thus there can be only four of them and, by the symmetry of the conic with regard to the axes, they lie pair and pair on two diameters equally inclined to the axes. Thus CL, CM are equally inclined to the axes, and the common chords PQ, RS of the conic and circle are equally inclined to the axes.

The same holds of the other pairs of common chords, viz. PS, RQ; PR, QS.

In particular, if a circle and conic have three coincident intersections P, Q, R, the common chord PS and the common tangent at P are equally inclined to the axes.

The circle is then said to be the *circle of curvature at* P: this property enables us to construct it graphically.

87. Every ellipse can be derived from a circle by an orthogonal projection. For consider the orthogonal projection of a circle upon any plane through a diameter x. A line perpendicular to this axis of collineation x is still perpendicular to x after projection and rabatment about x into the original plane, and if P be any point of the original figure, PX the perpendicular from P on x, P' the corresponding point

of the rabatted projection, P' lies on PX and $P'X : PX$ = cosine of dihedral angle θ between the two planes. Thus the pole of perspective is at infinity in the direction perpendicular to x. The perspective relation between the circle and ellipse figures is equivalent to a *stretch*, the stretch-ratio being $\cos\theta$.

Also because the projection is cylindrical the lines at infinity correspond : therefore their poles, that is, the centres, correspond. It follows that conjugate diameters of the circle project into conjugate diameters of the ellipse : in particular x and the perpendicular diameter of the circle, being conjugate diameters, project into perpendicular conjugate diameters of the ellipse, since a perpendicular to x remains perpendicular to x. These give the axes of the ellipse. If a be the radius of the circle, a and $a\cos\theta$ are the major and minor semi-axes of the ellipse.

Conversely an ellipse of semi-axes a and b ($a>b$) can be obtained in this manner by projecting a circle of radius a orthogonally upon a plane making with the plane of the circle an angle

$$\theta = \cos^{-1}\left(\frac{b}{a}\right).$$

An ellipse being completely given by its principal axes (see Art. 69), it follows that every ellipse can be obtained in this way from a circle on its major axis as diameter.

The circle on the major axis of an ellipse as diameter is called its *auxiliary circle*. Thus the ellipse and its auxiliary circle are derivable one from the other by a stretch parallel to the minor axis.

88. The conjugate parallelogram. A *conjugate parallelogram* is one whose sides are the tangents at the extremities of two conjugate diameters. Clearly no real conjugate parallelogram can exist, except in the case of the ellipse.

Consider the ellipse as defined by the stretch of its auxiliary circle. Since by Art. 87 conjugate diameters correspond to conjugate diameters and parallel lines to parallel lines (because the vanishing lines are at infinity) it follows that a conjugate parallelogram for the ellipse corresponds to a conjugate parallelogram for the circle.

But a conjugate parallelogram for a circle is a circumscribed square, because conjugate diameters of a circle are at right angles.

Now, in any stretch, corresponding areas are to one another in the stretch ratio. For consider an elementary parallelogram

$PQRS$ (Fig. 30) of which the sides PQ, RS are parallel to the stretch axis and the sides PS, QR are parallel to the direction of stretch. Let these meet the stretch axis at X, Y respectively. $PQRS$ transforms into a parallelogram $P'Q'R'S'$ of which $P'Q'$, $R'S'$ are parallel to the stretch axis. For if λ be the stretch ratio

$$P'X = \lambda \,.\, PX = \lambda \,.\, QY = Q'Y \text{ and } S'X = R'Y = \lambda \,.\, SX.$$

Hence $$P'S' = \lambda\,(XS - XP) = \lambda \,.\, PS$$

and

parallelogram $P'Q'R'S'$: parallelogram $PQRS = S'P' : SP = \lambda$.

Breaking up any area into such elementary parallelograms

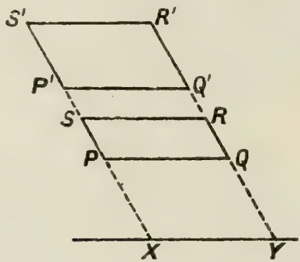


Fig. 30.

and adding we see that λ = ratio of two corresponding areas.

Hence area of any conjugate parallelogram of the ellipse

$= \dfrac{b}{a} \times$ area of corresponding circumscribed square of the auxiliary circle

$= \dfrac{b}{a} \,.\, 4a^2 = 4ab.$

Thus the conjugate parallelogram of an ellipse is of constant area. Calling p the perpendicular from the centre on any given tangent, d the length of the semi-diameter parallel to this tangent, the area of the conjugate parallelogram of which the given tangent is a side is clearly $4pd$, for $2d$ is the base of the parallelogram and $2p$ is the height. $\therefore\ 4pd = 4ab$, i.e. $pd = ab$.

89. **Sum of squares of two conjugate semi-diameters.** Let CP', CQ' (Fig. 31) be two diameters at right angles of the auxiliary circle, CP, CQ the corresponding diameters

of the ellipse. Then CP, CQ are conjugate. Let $P'PM$, $Q'QN$, be perpendicular to the major axis.

Then from the stretch property, if ϕ be the angle ACP',

$$PM = \frac{b}{a}P'M = \frac{b}{a}a\sin\phi = b\sin\phi;$$

$$QN = \frac{b}{a}Q'N = \frac{b}{a}a\cos\phi = b\cos\phi.$$

$$CM = a\cos\phi; \quad CN = a\sin\phi;$$

$$\begin{aligned} CP^2 + CQ^2 &= CM^2 + PM^2 + QN^2 + CN^2 \\ &= a^2\cos^2\phi + b^2\sin^2\phi + b^2\cos^2\phi + a^2\sin^2\phi \\ &= a^2 + b^2. \end{aligned}$$

Hence the sum of the squares of two conjugate diameters of an ellipse is constant.

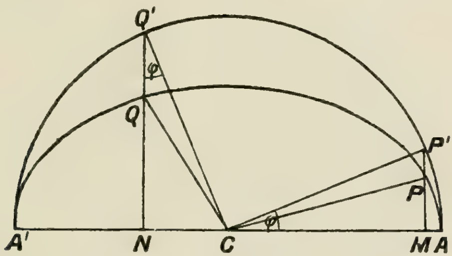


Fig. 31.

90. Pseudo-conjugate parallelogram. Let ACA' (Fig. 32) be a diameter of a hyperbola meeting the curve at real points A, A'. Let the tangent at A meet the asymptotes at D, E and the tangent at A' meet them at F, G. Then, because the tangents at A, A' are parallel, $DEFG$ is a parallelogram of which the asymptotes are diagonals.

If P be a point on the curve and PN the chord through P conjugate to ACA' meeting ACA' at N,

$$\frac{PN^2}{AN \cdot NA'} = -\frac{CB_1^2}{CA^2},$$

CB_1 being the real length (see Art. 84) of the diameter conjugate to ACA'.

But $$\frac{PN^2}{AN \cdot NA'} = -\left(\frac{AN}{A'N}\right)\left(\frac{PN}{AN}\right)^2.$$

If P moves off to infinity on the hyperbola in the direction of the asymptote CD, AP becomes parallel to CD. The triangles CAD, ANP become similar, and $\left(\frac{PN}{AN}\right)^2$ becomes equal to $\left(\frac{AD}{CA}\right)^2$; also $\frac{AN}{A'N}$ approaches unity.

Hence $$\frac{CB_1^2}{CA^2} = \frac{AD^2}{CA^2},$$

or $$CB_1 = AD.$$

Thus the intercept of a tangent between the asymptotes measures the real length of the parallel diameter. The parallelogram $DEFG$ is therefore a *pseudo-conjugate* parallelogram. Its

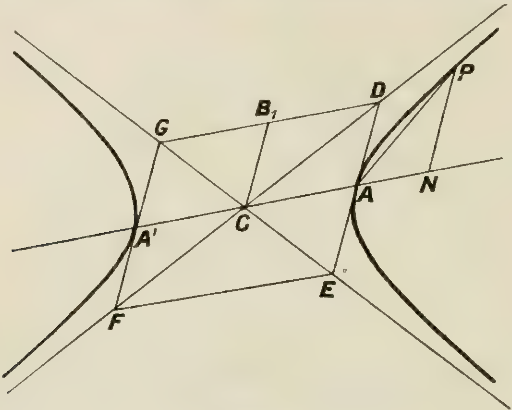


Fig. 32.

median lines are conjugate diameters, but only one pair of sides touches the curve.

We have seen (Art. 75) that the area of the triangle ECD cut off from the asymptotes by any tangent to the curve is constant. The area of the pseudo-conjugate parallelogram $DEFG$ is four times the area of the triangle ECD and is therefore also constant.

If p = perpendicular from C on tangent at A, $d = CB_1$ = real length of diameter conjugate to CA, $DE = 2d$, and area of triangle $ECD = \frac{1}{2}p \cdot 2d = pd$. Hence in the hyperbola as in the ellipse pd = constant. Taking the case where the sides of the

pseudo-conjugate parallelogram are parallel to the axes, the constant $= ab$ where a is the semi-transverse axis and b is the real length of the semi-conjugate axis.

91. Difference of squares of real lengths of conjugate semi-diameters of a hyperbola. From the triangle CED (Fig. 32) we have, since CA is the median,

$$CD^2 + CE^2 = 2(CA^2 + AD^2)$$

by a well-known relation.

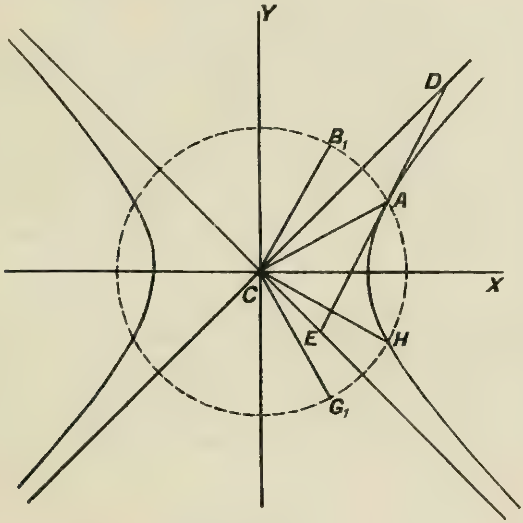


Fig. 33.

Also $\quad DE^2 = CE^2 + CD^2 - 2CE \,.\, CD \cos ECD,$

that is, $\quad 4AD^2 = 2CA^2 + 2AD^2 - 2CE \,.\, CD \cos ECD,$

or $$CA^2 - AD^2 = CE \,.\, CD \,.\, \cos ECD.$$

But $CE \,.\, CD$ is constant since the area of the triangle ECD is constant. Hence, the angle ECD being likewise constant,

$$CA^2 - AD^2 = \text{constant}.$$

The constant is easily seen to be $a^2 - b^2$ by taking CA, AD parallel to the transverse and conjugate axes.

92. Rectangular hyperbola. If the angle between the asymptotes is a right angle the curve is called a rectangular hyperbola.

Conjugate diameters of a rectangular hyperbola are equal: For if DE (Fig. 33) be the tangent at A to such a hyperbola, meeting the asymptotes at D and E; since the angle at C is a right angle, the circle on DE as diameter passes through C. Hence $AC = AD$. Also CA, AD, i.e. CA, CB_1 are then equally inclined to the asymptotes (CB_1 having the same meaning as in Art. 90).

In particular the transverse and conjugate semi-axes are equal for a rectangular hyperbola.

Again consider the semi-diameter CH perpendicular to CB_1. Because CE, CH are perpendicular to CD, CB_1 respectively, the angle ECH = angle DCB_1 = angle ACD. CA, CH are thus equally inclined to the axes CX, CY and CH is therefore real and equal to CA, that is to CB_1. Thus the real lengths of perpendicular semi-diameters of a rectangular hyperbola are equal.

The diameter CG_1 conjugate to CH makes the angle G_1CE = angle ECH = angle ACD. It is therefore perpendicular to CA. We have thus a set of four diameters equal in real length.

Notice that this does not invalidate the result mentioned in Art. 86 that only one diameter exists equal to a given diameter; and that this diameter is equally inclined to the axes with the given diameter. For the lengths CB_1, CG_1 are not semi-diameters at all, but merely the analogues of semi-diameters: they are only called such by a convention, B_1, G_1 not being points on the curve.

EXAMPLES Va.

1. Show that by altering the order of six points on a conic, 60 different hexagons may be formed, with 60 corresponding Pascal lines.

Show that these 60 hexagons have their Pascal lines concurrent in fours, namely when they have a pair of opposite sides common.

2. Show that in the notation of Art. 71 for Pascal's Theorem the lines (13, 46), (35, 62), (51, 24) are concurrent.

3. Show that (13, 46), (35, 62), (51, 24) are possible Pascal lines for the six points.

4. State and prove the results corresponding to those of Exs. 1, 2, 3 for Brianchon's Theorem.

5. Prove that if ABC, $A'B'C'$ be two triangles inscribed in a conic, their six sides touch another conic.

[The sides BC, CA, AB are denoted by a, b, c; $B'C'$, $C'A'$, $A'B'$ by a', b', c'.

We have

$$B(ACA'C') \barwedge B'(ACA'C'), \quad \therefore B(ACA'C') \barwedge B'(C'A'CA).$$

Cut these by $A'C'$, AC. Then by the theorem of the cross-axis

$$(bc', b'c), (ca', c'a), (ab', a'b)$$

are concurrent. By the converse of Brianchon's Theorem the result follows.]

6. Deduce from Ex. 5 Poncelet's Theorem that if there exist one triangle inscribed in one conic and circumscribed to another, there exist an infinity of such triangles.

7. Given five points A, B, C, D, E on a conic, construct the tangent to the conic at any one of them.

8. Given five tangents a, b, c, d, e to a conic, construct the point of contact of any one of them.

9. Deduce the results of Art. 75 from the property that a pair of conjugate diameters of a hyperbola are harmonically conjugate with regard to the asymptotes, without using Pascal's Theorem.

10. Obtain the theorem that a variable tangent to a hyperbola cuts off from the asymptotes a triangle of constant area by applying Brianchon's Theorem to the hexagon $aapbbq$, a, b being the asymptotes, p, q any two tangents.

11. Obtain the following construction for a parabola by tangents, given four tangents a, b, c, d. On (ab, cd) take any point B. Draw through B a parallel to a meeting c at P, and a parallel to d meeting b at Q. PQ is a tangent to the parabola.

12. Show how to construct a parabola by tangents given three tangents and the direction of the axis. Construct also the vertex and axis.

13. Prove the following construction for a parabola, given a point A, the tangent a at A, a parallel x to the axis and another point B on the curve. Construct a parallelogram $ACBD$ with AB as diagonal and BD, BC parallel to a, x respectively. Draw LM parallel to the diagonal CD to meet BC at L, BD at M. The meet of AL and a parallel to x through M is a point on the curve.

14. Deduce the construction of Ex. 13 from that of Ex. IV A, 22, by taking B at infinity.

15. From the constructions of Art. 80 deduce that MQ' is proportional to $(AQ')^2$.

16. If the tangent at U meet a pair of conjugate diameters at P, P', show that $UP \,.\, UP' = CD^2$ where CD is the diameter parallel to the tangent at U.

17. Show how to find graphically the directions of the axes of any given conic without first finding the centre.

18. Show that if a circle touch a conic and cut it again in two real points the common chord and the common tangent are equally inclined to the axes.

19. If a circle and conic have double contact the common chord of contact is parallel to an axis.

20. Show that if P, P' be points where a perpendicular to the major axis of an ellipse meets the curve and the auxiliary circle respectively the tangents at P, P' meet on the major axis.

21. Prove that an ellipse can be obtained from the circle on its minor axis as diameter by a stretch parallel to the major axis.

22. Show that the area of an ellipse is πab.

23. Show that the diagonals of a conjugate parallelogram are themselves conjugate diameters.

24. Show how to construct geometrically the equal conjugate diameters of an ellipse, given its axes.

25. Show that if a parallelogram be circumscribed to or inscribed in a conic its diagonals intersect at the centre of the conic.

26. The extremities A, A' of a diameter of a rectangular hyperbola are joined to a point P on the curve. Show that AP, $A'P$ describe two oppositely equal pencils. Show also that this is not true unless A, A' are extremities of a diameter.

27. Prove that the diagonals of a complete quadrilateral circumscribed about a conic are divided harmonically by the diagonal points of the complete quadrangle formed by their four points of contact.

28. Show that the triangle formed by three tangents to a conic is in plane perspective with the triangle formed by their three points of contact.

29. The tangents to a hyperbola at P and Q meet an asymptote at R, S respectively. If $(RP, SQ) = U$, $(SP, RQ) = V$, show that UV is parallel to this asymptote.

30. If a straight line meet a hyperbola at P and the asymptotes at Q, R, prove that $PQ . PR =$ square of parallel semi-diameter.

31. Show that any chord of a rectangular hyperbola subtends equal or supplementary angles at the extremities of any diameter.

EXAMPLES V B.

1. Given that the angle between the axes of coordinates is $75°$, draw the hyperbola having the axes for asymptotes and passing through the point $(2, \cdot 5)$.

2. Draw the conic through the points $(1, 1\cdot 5)$, $(4, \cdot 5)$, $(5\cdot 8, 4\cdot 3)$, $(6, 2)$, $(3, 4\cdot 4)$ by the Pascal line method.

3. A conic passes through the points whose rectangular coordinates are $(2, 0)$, $(\frac{5}{2}, \frac{7}{2})$, $(4, 2)$, and touches the line $4y = 7(x+1)$ at the point $(\frac{5}{7}, 3)$. Draw the conic.

4. Draw the parabola which touches four sides of a regular pentagon inscribed in a circle of $2''$ radius.

5. A hyperbola passes through the points $(1, 0)$, $(1, 3)$, $(6, 0)$, $(6, 4)$, and has one asymptote parallel to $y = 2x$. Construct both asymptotes in position.

6. A hyperbola has the axis of y for one asymptote and touches the lines $x + y = 4$, $\frac{x}{5} + \frac{y}{2} = 1$, $\frac{x}{3} + \frac{y}{8} = -1$. Construct it by tangents.

7. A parabola touches the line $y = x$ at the origin and its axis is parallel to Ox. If it passes through the point $(4, -3)$, draw the curve, the axes of coordinates being rectangular.

8. Construct the vertex and axis of the parabola through the three points $(2, -1)$, $(3, 2)$, $(6, 4)$ whose axis is parallel to the line $14x + 3y = 21$.

9. Draw the parabola which touches three sides AB, BC, CD of a regular pentagon $ABCDE$ of side $1''$ and whose axis is parallel to the line joining C to the middle point of AB. Construct its axis and vertex.

CHAPTER VI.

FOCAL PROPERTIES OF THE CONIC.

93. **Every conic can be obtained as the section of a real right circular cone.** Describe a circle (Fig. 34 a) touching the conic at the extremity X of an axis XA'. This axis is therefore a diameter of both conic and circle. The conic and circle may then be brought into plane perspective by taking x as axis of collineation (see Art. 43). The pole O of perspective is then by symmetry on the axis XA'. The vanishing line i of the circle is a parallel to x, cutting the axis of symmetry at I. If we rotate the conic about x, and join corresponding points of the rotated conic and the circle, these joins, by Art. 12, will pass through a vertex V which lies on a circle centre I and radius IO in a plane through the axis of symmetry perpendicular to the plane of the paper. This circle will therefore in general intersect the perpendicular to the plane of the paper through the centre C of the given circle. Take for V one of these intersections. Then the projection of the given circle from V upon a plane through x parallel to Vi is the given conic. The given conic is $\therefore$ the section by this plane of a cone vertex V and base the given circle. But this is a right circular cone, since V is on the perpendicular to the plane of the given circle through its centre.

It remains to show that the dimensions of the given circle can be so adjusted as to make this construction real in all cases.

In order that the circle locus of V should cut the perpendicular through C in real points it is necessary and sufficient that $IO > IC$.

Ellipse. In the case of the ellipse take X the extremity of the major axis and take for the circle a radius intermediate between the major and the minor semi-axes of the ellipse. The figure is then as drawn in Fig. 34 a. The common tangents must necessarily intersect at O on the side of C further from X.

Also if I'^{∞} be the point at infinity on XC, the tangent to the ellipse from I'^{∞} meets the tangent to the circle from I at a point Y on x. XY = semi-minor axis of the ellipse, and this being less

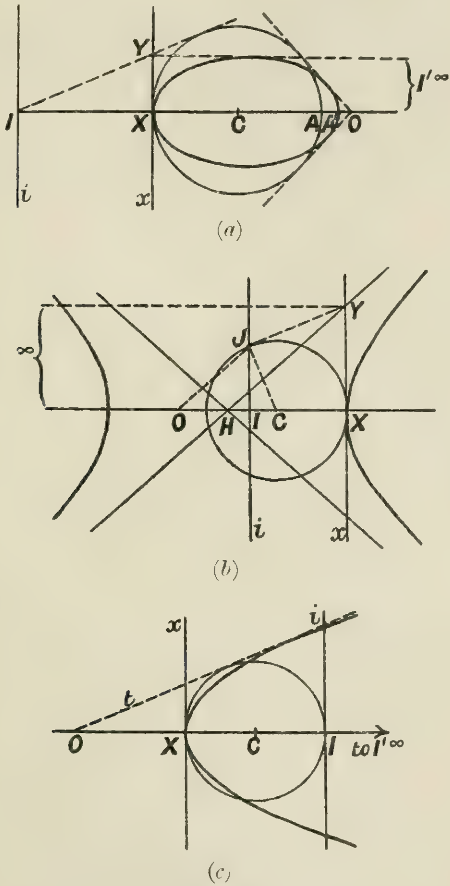


Fig. 34.

than radius of circle, the tangent YI to the circle meets CX at I on the same side of C as X. Thus C lies between I and O and the construction leads to a real right circular cone.

Hyperbola. Take X an extremity of the transverse axis (Fig. 34 *b*). Let x meet one asymptote at Y; let J'^{∞} be the point of contact of this asymptote. Then J is the point of contact of the tangent from Y to the circle. i is therefore the parallel to x through J, meeting XC at I. O is where JJ'^{∞}, i.e. the parallel through J to the asymptote, meets XC. $IO > IC$ if angle $OJI >$ angle CJI. But, H being centre of hyperbola, angle $OJI =$ angle HYX and because JI, JC are perpendicular to YI'^{∞}, YJ respectively, angle $CJI =$ angle JYI'^{∞}. Hence $IO > IC$ if the angle $HYX >$ angle JYI'^{∞}. Draw therefore YJ making with YI'^{∞} an angle less than XYH: that circle touching YJ and touching XY at X, whose centre falls on the same side of x as H, leads to a real solution.

Parabola. X being the vertex, x the tangent at X, draw any tangent t to the parabola meeting the axis of the curve at O. Describe a circle touching t and touching x at X, to fall on the same side of x as the parabola (Fig. 34 *c*). Then I is where the circle meets the axis of the parabola (for the vanishing line touches the circle). I, C fall inside the parabola, O, being on a tangent to the curve, falls outside it. Hence C must lie between I and O.

94. **Focal spheres.** Consider a right circular cone vertex V. Let it be cut by any plane in a conic s and let the plane of the paper (Fig. 35) be the plane through V perpendicular to the plane of section. By symmetry the line AB in which the plane of the paper meets the plane of section is an axis of the section.

Construct a sphere touching the plane of section at S and touching the cone. It will touch the cone along a circle s' of which $B'A'$ (Fig. 35) is a diameter. The plane of this circle meets the plane of section in a line x perpendicular to the plane of section and meeting the plane of the paper at X.

Considering the circular section $B'A'S$ of this sphere by the plane of the paper we note that: X lies on the polar of V with regard to this circle and X lies on the polar of S (this being the tangent at S). Hence X is the pole of SV or X, S' are harmonically conjugate with regard to B', A'. Therefore x is the polar of S' with regard to the circle s' and by projection x is also the polar of S with regard to the conic s. x being an axis of collineation, if on x we take two points P, Q which are conjugate with regard to s' they are also conjugate with regard to s. But because they are conjugate with regard to s', $\therefore$ by Art. 58, $PX \cdot XQ =$ square of tangent from X to circle $s' =$ square of tangent from X to sphere $= XS^2$. It follows by a well-known result (XS being perpendicular to PQ) that the angle PSQ is a right

angle. And P, Q being conjugate points on the polar of S, SP, SQ are conjugate lines with regard to the conic s.

Hence S is a point such that every pair of conjugate lines through it are at right angles. A point possessing this property is called a *focus* of the conic.

Clearly one other sphere can in general be drawn touching the cone and the plane. If H be its point of contact with the plane, H is also a focus of the conic.

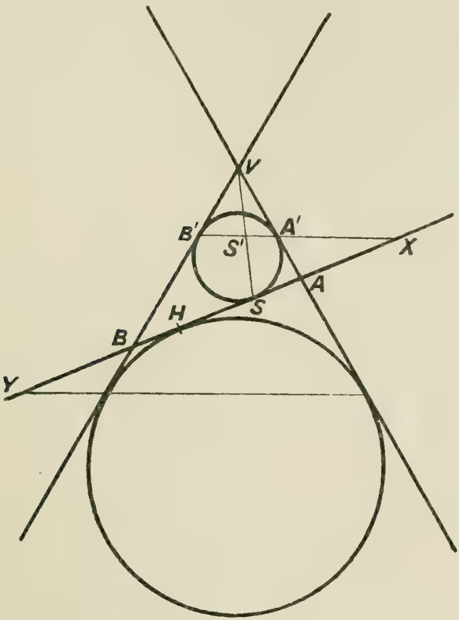


Fig. 35.

These two spheres are called the *focal spheres*.

The lines x, y in which the planes of contact of the focal spheres with the cone meet the plane of section are polars of the foci S, H and are termed the *directrices* of the conic.

95. Foci. There cannot be more than two real foci; for if S be one focus, x the corresponding directrix, let F be any other focus. Join FS. Let the perpendicular through S to SF

meet x at R. Because SF, SR, being perpendicular, are conjugate lines through S, therefore the pole of SF lies on SR: also it lies on the polar x of S. $\therefore$ R is the pole of SF and FS, FR are conjugate lines through F. But these cannot be at right angles unless R is at infinity, that is, unless F is on the perpendicular through S to x, i.e. on the axis AB.

Also there cannot be three foci on the axis AB. For consider two points U^∞, V^∞ at infinity in two directions at right angles in the plane. Let conjugate rays through U^∞, V^∞ meet the axis AB at T, T'. The pencils $U^\infty[T]$, $V^\infty[T']$ are projective by Art. 55 and $\therefore$ the ranges $[T]$, $[T']$ are projective.

But the lines joining a focus to U^∞, V^∞, being perpendicular lines through a focus, are conjugate lines through U^∞, V^∞. Hence at a focus the points T, T' coincide. Foci are therefore self-corresponding points of the ranges $[T]$, $[T']$, and as these ranges by Art. 25 cannot have more than two self-corresponding points, there can be only two real foci; and these are the points S, H found in the preceding article.

Also from the symmetry of the curve about the axes it follows that the two foci S, H and the two corresponding directrices x, y are symmetrically situated with regard to the centre and the axis perpendicular to AB.

Also the foci are always *inside* the curve, for if they were outside, the tangents from the foci would be self-conjugate, or perpendicular to themselves, which is impossible.

In the parabola the vertex B (Fig. 35) goes off to infinity. One focal sphere then goes to infinity and there is only one focus at a finite distance.

In what follows the axis AB, on which the real foci lie, will usually be spoken of as the *focal axis* of the curve. From the constructions of Arts. 93, 94 it appears that the focal axis is the major axis in the ellipse and the transverse axis in the hyperbola.

96. Eccentricity. Let P, Q (Fig. 36) be two points on the conic, S a focus, XY the corresponding directrix. Let PQ meet the directrix at Y and let PM, QN be perpendiculars from P, Q on the directrix. Then if the tangents at P and Q meet at T, T is the pole of PQ, S is the pole of XY, $\therefore$ Y is the pole of TS, $\therefore$ SY, ST are conjugate lines through a focus, $\therefore$ they are at right angles. Also ST being the polar of Y, it follows that Y, P, the point where ST meets PQ, and Q are four harmonic points. Hence $S(YPTQ)$ is a harmonic pencil and the conjugate rays SY, ST being perpendicular, bisect the angles between the other two (Art. 28).

$$\therefore\ |YP|:|YQ| = |SP|:|SQ|.$$

But by similar triangles

$$|YP|:|YQ| = |PM|:|QN|$$

or $$|SP|:|SQ| = |PM|:|QN|,$$

i.e. $$|SP|:|PM| = |SQ|:|QN|$$

$$= \text{constant for the conic} = e \text{ say; hence:}$$

The distance of a point on a conic from a focus is to its distance from the corresponding directrix in a constant ratio e, which is called the eccentricity.

From the symmetry of the curve it is obvious that the eccentricity must be the same for the two foci.

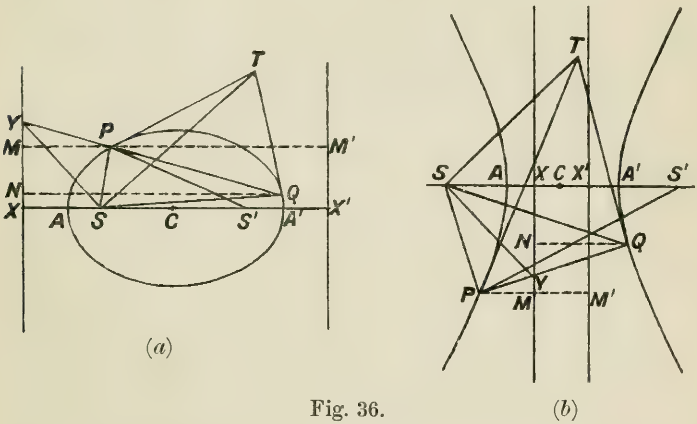


Fig. 36.

97. Distances of foci and directrices from the centre. If the focal axis meet the curve at A, A' and S, S' be the foci (Fig. 36), then A and A' are the points which divide SX in the ratio $e:1$. If therefore $e=1$, A' is at infinity and we have a parabola: if $e<1$, S (and therefore by symmetry S') is inside AA' as in Fig. 36a, hence the centre C, which is the mid-point of SS', is inside the curve and the latter is an ellipse. If $e>1$, S and S' are outside AA'. The centre C is on the opposite side of the curve to the foci; it is therefore outside the curve and the latter is a hyperbola.

We have, since A, A' divide SX internally and externally in the ratio $e:1$,

$$SA = e \,.\, AX,$$

$$SA' = -e \,.\, A'X = e \,.\, AX' \text{ by symmetry.}$$

Subtracting, $$AA' = e \,.\, XX'$$

or $$CA = e \,.\, CX \qquad\qquad (1).$$

But since $(A'SAX)$ is harmonic,

$$CS \,.\, CX = CA^2 \qquad\qquad (2).$$

Hence using (1) $$CS = e \,.\, CA \qquad\qquad (3).$$

98. **Tangents from an external point subtend equal or supplementary angles at the focus.** By Art. 96 SY, ST (Fig. 36) bisect the angles between SP, SQ. If P and Q are on the same branch of the curve (Fig. 36 a), Y lies outside PQ; $\therefore$ ST is the internal bisector of the angle PSQ and the two tangents TP, TQ subtend equal angles at the focus. If P and Q are points on opposite branches of the curve (Fig. 36b), Y lies inside PQ; $\therefore$ ST is the external bisector of the angle PSQ and $TSQ + TSP =$ twice (mean angle TSY) $= 2$ right angles. Hence the tangents TP, TQ subtend supplementary angles at the focus.

Incidentally we have proved the following theorem: if a chord PQ meet the directrix corresponding to a focus S at Y, then SY bisects internally or externally the angle PSQ according as P, Q lie on opposite branches or on the same branch of the conic.

If P and Q coincide, T coincides with them. Hence the intercept on a tangent to the curve between the point of contact and a directrix subtends a right angle at the corresponding focus.

If a, b be two fixed tangents to the conic, c a variable tangent meeting a, b at L, M respectively, S a focus, A, B, C the points of contact of a, b, c, and if a, b, c all touch the same branch of the curve, we have by the above $LSC = \frac{1}{2} ASC$, $CSM = \frac{1}{2} CSB$.

Hence by addition $LSM = \frac{1}{2} ASB$. Therefore LM subtends a fixed angle at S.

The student may prove for himself that if a, b, c touch different branches of the curve the theorem still holds, but that LSM is then equal to $\pi - \frac{1}{2} ASB$, $\frac{\pi}{2} - \frac{1}{2} ASB$ or $\frac{\pi}{2} + \frac{1}{2} ASB$ according to circumstances.

99. **Sum and difference of focal distances.** Through

any point P of the conic draw a parallel to the focal axis meeting the directrices at M, M'. Then in Fig. 36a (ellipse)

$$|SP|+|S'P|=e\{|PM|+|PM'|\}=e|MM'|=e|XX'|=AA',$$

or the sum of the focal distances = major axis.

Similarly in Fig. 36b (hyperbola)

$$|S'P|\sim|SP|=e\{|PM'|-|PM|\}=e|MM'|=AA',$$

or the difference of the focal distances = transverse axis.

100. Semi-latus rectum. Length of axis perpendicular to focal axis.

The chord LL' (Fig. 37) through the focus perpendicular to the focal axis is called the *latus rectum* of the conic.

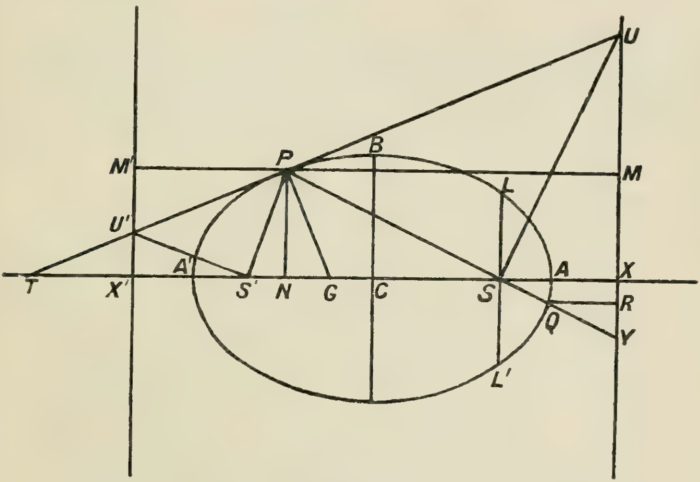


Fig. 37.

We have, since SX = perpendicular from L on corresponding directrix,

$$\text{Semi-latus rectum} = SL = e\,.\,SX$$
$$= e(CX - CS)$$
$$= e\left(\frac{CA}{e} - e\,.\,CA\right) = CA(1-e^2)\ldots(1).$$

By Newton's theorem

$$\frac{SL^2}{A'S\,.\,SA} = \frac{CB^2}{CA^2},$$

where CB = semi-axis perpendicular to AA'.

But $\quad A'S = CS - CA' \qquad SA = CA - CS$

$$= CS + CA$$

$$= (e+1)\,CA \qquad SA' = CA\,(1-e) \quad \text{.........(2)},$$

$$A'S\,.\,SA = CA^2\,(1-e^2),$$

$$\therefore\ \frac{CB^2}{CA^2} = \frac{CA^2\,(1-e^2)^2}{CA^2\,(1-e^2)} = 1 - e^2 \quad \text{..................(3)},$$

$$\therefore\ CB^2 = (1-e^2)\,CA^2 = A'S\,.\,SA = CA\,.\,|SL| \quad \text{......(4)}.$$

If $e > 1$, CB^2 is negative. If CB_1 be the real length of the conjugate semi-axis

$$CB_1^2 = (e^2-1)\,CA^2 = SA\,.\,SA' = CA\,|SL|$$

as before.

101. The semi-latus rectum is a harmonic mean between the segments of any focal chord. Let PSQ (Fig. 37) be a focal chord, meeting the directrix at Y. Then $\because$ $(YQSP)$ is harmonic, we have by Art. 28

$$\frac{1}{YP} + \frac{1}{YQ} = \frac{2}{YS}.$$

But $\quad YP : YQ : YS = PM : QR : SX$

$$= e\,.\,PM : e\,.\,QR : e\,.\,SX$$

$$= PS : SQ : SL.$$

Hence $$\frac{1}{PS} + \frac{1}{SQ} = \frac{2}{SL}.$$

Multiplying up

$$\frac{PS + SQ}{PS\,.\,SQ} = \frac{2}{SL} \text{ or } \frac{PQ}{PS\,.\,SQ} = \frac{2}{SL}.$$

If CD be the diameter parallel to PQ, CB is the diameter parallel to SL. Hence by Newton's theorem

$$\frac{PS\,.\,SQ}{SL^2} = \frac{CD^2}{CB^2}.$$

Thus $|PQ| = 2\,|SL|\,.\,\dfrac{CD^2}{|CB|^2} = 2\,\dfrac{CD^2}{CA}$ by Art. 100.

Hence the lengths of focal chords are proportional to the squares of the parallel semi-diameters.

102. The tangent and normal bisect the angles between the focal distances. Let P (Fig. 37) be a point on the conic and let the tangent at P meet the directrices at

U, U'. Let the parallel through P to the focal axis meet the directrices at M, M'.

Then

$$|SP|:|S'P| = e\,|PM|:e\,|PM'| = |PM|:|PM'|$$
$$= |PU|:|PU'|,$$

and in the triangles USP, $U'S'P$ the angles at S, S' are right angles. Hence these triangles are similar and the angle SPU = angle $S'PU'$.

Hence the tangent and normal at P bisect the angles between the focal distances.

The above proof holds formally for the hyperbola as well as for the ellipse (the student, however, is advised to draw the figure for the hyperbola and compare).

In the ellipse the points of the focal axis outside the curve are outside SS'. Hence the point T where the tangent at P meets the focal axis, being a point on a tangent, is outside the curve and $\therefore$ outside SS'. Therefore in an ellipse the tangent bisects externally and the normal bisects internally the angle SPS'.

On the other hand in the hyperbola the points of the focal axis outside the curve lie inside SS'. T is therefore inside SS'. Therefore in a hyperbola the tangent bisects internally and the normal bisects externally the angle SPS'.

103. Intercept of the normal on the focal axis. Let the normal at P (Fig. 37) meet the focal axis at G. Because PG, PT bisect SPS', $\therefore$ S, G, S', T are harmonic,

$$\therefore\ CG\,.\,CT = CS^2 \quad \text{.....................(1).}$$

If PN be the perpendicular from P on the focal axis, PN goes through the point of contact of the symmetrical tangent through T. Thus PN is the polar of T and A, N, A', T are harmonic,

$$\therefore\ CN\,.\,CT = CA^2 \quad \text{.....................(2).}$$

Dividing, $$\frac{CG}{CN} = \frac{CS^2}{CA^2} = e^2 \text{ or } CG = e^2.\,CN \quad \text{...........(3).}$$

Also since PG bisects the angle SPS', we have

$$\frac{|SG|}{|SP|} = \frac{|GS'|}{|PS'|} = \frac{SG + GS'}{SP + PS'} = \frac{SS'}{AA'} = e \text{ (for ellipse),}$$
$$= \frac{SG - S'G}{SP - S'P} = \frac{SS'}{AA'} = e \text{ (for hyperbola).}$$

Thus $$SG = e \, . \, SP \qquad\qquad (4).$$

104. The feet of focal perpendiculars on a tangent lie on the auxiliary circle. Let SY, $S'Y'$ (Fig. 38) be perpendiculars from S, S' upon the tangent at P. Let $S'P$ meet SY at F.

Consider the case where the conic is an ellipse.

The angles FPY, SPY are equal, for $FPY = S'PY' = SPY$ by Art. 102. Hence the angles FYP, SYP being right angles and YP being common the triangles FYP, SYP are congruent.

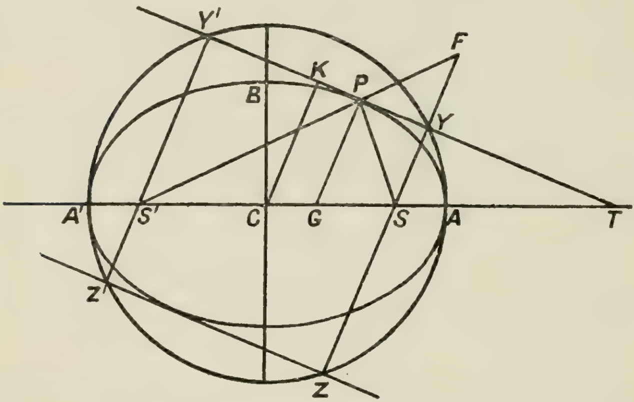


Fig. 38.

$\therefore |FP| = |SP|$ and $|FS'| = |SP| + |S'P| = 2|CA|$. Also C, Y being mid-points of SS', SF respectively, $|CY| = \frac{1}{2}|FS'| = |CA|$. Hence Y lies on the auxiliary circle. Similarly Y' lies on the auxiliary circle. The proof when the conic is a hyperbola is precisely similar and may be left as an exercise for the student.

If Z, Z' be the feet of perpendiculars upon the tangent parallel to the tangent at P, Z and Z' also lie on the auxiliary circle.

Also by symmetry

$$|SZ| = |S'Y'|; \quad |S'Z'| = |SY|.$$

Hence $SY \, . \, S'Y' = ZS \, . \, SY = A'S \, . \, SA$ by the property of segments of chords of a circle.

If the conic be an ellipse SY, $S'Y'$ are drawn in the same sense and $A'S \, . \, SA$ is positive and equal to CB^2 (Art. 100).

Hence $$SY \,.\, S'Y' = CB^2.$$

If the conic be a hyperbola SY, $S'Y'$ are drawn in opposite senses and $A'S \,.\, SA$ is negative and equal to $-CB_1^2$ (Art. 100).

Hence $$SY \,.\, Y'S' = CB_1^2.$$

105. The normal PG is inversely proportional to the perpendicular CK from the centre on the tangent at P. For since $(TSGS')$ (Fig. 38) is a harmonic range (Art. 103),

$$\frac{1}{TS} + \frac{1}{TS'} = \frac{2}{TG}.$$

But $$TS : TS' : TG = SY : S'Y' : GP,$$

$$\therefore \frac{1}{SY} + \frac{1}{S'Y'} = \frac{2}{GP},$$

$$\frac{SY + S'Y'}{SY \,.\, S'Y'} = \frac{2}{GP}.$$

But $SY + S'Y' = 2CK$ because C is the middle point of SS' and $SY \,.\, S'Y' = CB^2$ by Art. 102,

$$\therefore \frac{2CK}{CB^2} = \frac{2}{GP},$$

or $$CK \,.\, GP = CB^2.$$

106. Special properties of the parabola. Most of the special properties of the parabola are deduced at once from those of the ellipse and the hyperbola by removing one vertex A' and the corresponding focus S' to infinity. The line $S'P$ then becomes a parallel to the axis.

The theorem of Art. 102 then becomes:

The tangent and normal to a parabola at a point P are equally inclined to the axis and to the focal distance, the tangent bisecting internally the angle SPM. It follows that, with the notations of Arts. 100—105,

$$|ST| = |SP| = |SG| = |PM| = |NX|.$$

Hence from $|SG| = |NX|$ we have $|NG| = |SX| =$ constant. Or the subnormal NG is constant in the parabola, and equal to $2|AS|$. For the eccentricity being unity, $|SA| = |AX|$ or $|SX| = 2|AS|$.

The circle on AA' as diameter becomes the straight line through A perpendicular to the axis, i.e. the tangent at the vertex.

The theorem of Art. 104 now reads:

The foot Y of the perpendicular from the focus upon any tangent to a parabola lies on the tangent at the vertex.

Again if in the theorem stated in the second part of Art. 98 we take c to be the line at infinity, then L, M are the points at infinity on a, b and the constant angle LSM is the angle between the two tangents. Hence S and ab subtend the same angle at L, M: $\therefore$ L, M, S, ab are concyclic or: the circle circumscribing the triangle formed by three tangents to a parabola passes through the focus.

Also the angle made by the tangent with the axis is half the angle made with the axis by the focal distance SP of its point of contact. Hence if TP, TQ be two tangents the angle between them is $\frac{1}{2}PSQ$, or by Art. 98 it is TSP or TSQ. Thus the angle between two tangents is equal to the angle subtended by either of them at the focus.

If this angle be a right angle, then the tangents are tangents at the extremities of a focal chord and T lies on the directrix. Therefore the directrix is the locus of intersections of perpendicular tangents to a parabola.

107. Parameter of parallel chords of a parabola. We have seen (Art. 84) that if QQ' be a chord of a parabola bisected at N by its conjugate diameter and if this diameter meet the curve at P, then

$$\frac{QN^2}{PN} = \text{a constant}.$$

This constant is called the parameter of the chords. To find its value take the chord QQ' to pass through the focus S (Fig. 39). If the diameter conjugate to QQ' meet the directrix at M, the pole of QQ' lies on this diameter and ($\because$ QQ' passes through S) on the directrix. Thus M is the pole of QQ'. MQ, MQ' are therefore tangents to the parabola and, M being on the directrix, these tangents are at right angles. The circle on QQ' as diameter passes through M.

$$\therefore\ |QN| = |NM| = 2\,.\,|PN|,$$

for M is the pole of QN and MN meets the curve at P and at infinity; MN is therefore divided harmonically by P and the point at infinity, that is, bisected at P.

Hence $$\frac{QN^2}{PN} = 4\frac{PN^2}{PN} = 4\,.\,PN.$$

But $$PN = MP = SP.$$

The parameter is therefore $4 . SP$.

108. The orthocentre of the triangle formed by three tangents to a parabola lies on the directrix. This is readily proved from Brianchon's theorem.

Let a, b, c be the three tangents, b', c' the two tangents perpendicular to b, c, and i^∞ the line at infinity.

Consider the hexagon $abb'i^\infty c'c$. Then $(ab, i^\infty c')$ (bb', cc') $(b'i^\infty, ca)$ are concurrent.

But $(ab, i^\infty c')$ is the parallel to c' through ab, i.e. the perpendicular from ab on c.

Similarly $(ac, i^\infty b')$ is the perpendicular from ac on b.

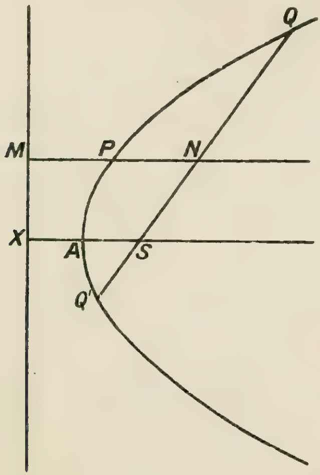


Fig. 39.

Hence the Brianchon point is the orthocentre of the triangle.

But (bb', cc') is the directrix, since perpendicular tangents b and b', c and c' meet on the directrix.

The orthocentre therefore lies on the directrix.

109. Circle of curvature. Suppose a circle to touch a conic at P and to meet it again at Q. If any line through Q meet the conic and circle again at R, R' respectively and the

tangent at P at T, we have by the well-known property of the circle

$$\frac{TP^2}{TQ \,.\, TR'} = 1,$$

and by Newton's theorem for the conic

$$\frac{TP^2}{TQ \,.\, TR} = \frac{\text{square of semi-diameter of conic parallel to } TP}{\text{square of semi-diameter of conic parallel to } TRR'},$$

or by division

$$\frac{TR'}{TR} = \frac{\text{square of semi-diameter parallel to tangent at } P}{\text{square of semi-diameter parallel to } TR}.$$

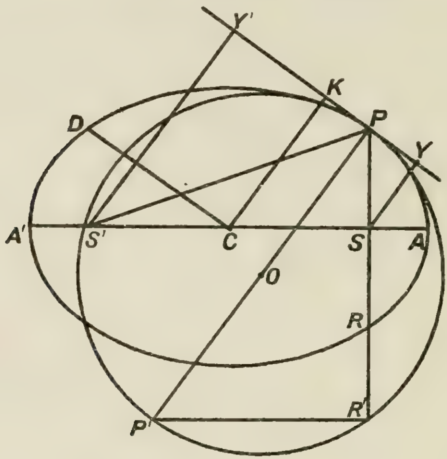


Fig. 40.

If we make Q coincide with P, the circle becomes the circle of curvature at P and T coincides with P. Hence if any chord through P (Fig. 40) be drawn to meet the conic at R and the circle of curvature at R',

$$\frac{PR'}{PR} = \frac{\text{square of semi-diameter parallel to tangent at } P}{\text{square of semi-diameter parallel to } PR}.$$

If the chord PR be drawn through a focus, we have

$$PR = \frac{2 \,.\, \text{square of semi-diameter parallel to } PR}{CA} \text{ by Art. 101.}$$

Hence

$$PR' = \frac{2\,.\,\text{square of semi-diameter parallel to tangent at } P}{CA} \ldots(1),$$

which gives the focal chord of curvature through P.

The form of the result (1) shows also that the focal chord of curvature through P is equal to focal chord of the conic parallel to the tangent at P.

When the conic is a parabola this focal chord is (see Fig. 39 and Art. 107) equal to $4\,.\,SP$.

This gives an easy construction for the radius of curvature of a parabola at P, namely: draw the perpendicular through the focus to SP, then four times the intercept this perpendicular makes on the normal at P is the diameter of the circle of curvature.

Returning to the case of the general conic, let PP' be the diameter through P of the circle of curvature at P (Fig. 40), Y, Y' the feet of the perpendiculars from S, S' on the tangent at P, then the right-angled triangles $PR'P'$, SYP, $S'Y'P$ are similar since the angles SPY, $S'PY'$ are equal by Art. 102 and the angles $PP'R'$, SPY are equal, both being complements of $P'PR$.

Hence $$\frac{PR'}{PP'} = \frac{SY}{SP} = \frac{S'Y'}{S'P} = \frac{SY + S'Y'}{SP + S'P} = \frac{2\,.\,CK}{2\,.\,CA},$$

CK being the perpendicular from the centre on the tangent at P.

But by equation (1) above, if CD (Fig. 40) be the semi-diameter parallel to the tangent at P,

$$PR' = \frac{2CD^2}{CA},$$

$$\therefore \frac{2CD^2}{CA\,.\,PP'} = \frac{CK}{CA},$$

and since $PP' = 2\,.\,OP$ where O is the centre of the circle of curvature,

$$\frac{CD^2}{OP} = CK,$$

or $$CK\,.\,OP = CD^2 \ldots\ldots\ldots\ldots\ldots\ldots(2).$$

But from Arts. 88, 90 we have $CK\,.\,CD = CA\,.\,CB$. Hence

$$OP = \frac{CD^3}{CA\,.\,CB} \ldots\ldots\ldots\ldots\ldots\ldots(3),$$

a fundamental formula for the radius of curvature at P.

EXAMPLES VI A.

1. Show that the construction of Art. 93 never leads to a real right circular cone if the circle touches the ellipse at the extremity of a minor axis.

2. On the transverse axis AB of a hyperbola as diameter a circle is drawn (the auxiliary circle of the hyperbola). A ray through A meets the circle and hyperbola in P, P'. Show that the tangents at P, P' meet on the tangent at B.

3. Prove that the focal axis of a plane section of a right circular cone is equal to the part of any generating line intercepted between its points of contact with the focal spheres, and that the perpendicular axis is a mean proportional between the diameters of the focal spheres.

4. Prove that the latus rectum of a plane section of a right circular cone is proportional to the perpendicular distance of the plane of section from the vertex of the cone.

5. Prove the following construction for the pole of any line q with regard to a conic, given the two foci S, S' and the two directrices s, s'. Let q meet s at P, s' at P'. Through S, S' draw perpendiculars to SP, $S'P'$ respectively: these meet at the point Q required.

6. Two points of a conic being given and also one of the directrices, show that the locus of the corresponding focus is a circle.

7. Two conics have a common focus. Prove that two of their common chords pass through the intersection of the directrices corresponding to this common focus.

8. A variable straight line meets two fixed straight lines at P, Q and PQ subtends a fixed angle at a fixed point S. Show that PQ touches a conic of which S is a focus.

9. Through a given point P two conics can be drawn having two given points S, H for foci. Of these one is an ellipse, the other a hyperbola, and they cut at right angles.

10. Prove that the orthogonal projection of the normal PG upon either focal distance is constant and equal to the semi-latus rectum.

[If K = foot of perpendicular from G on SP in Fig. 37, prove triangles PKG, USP similar and triangles SKG, UXS similar; and use $SG = e \,.\, SP$.]

11. If the normal at P meet the non-focal axis at G', the projection of PG' upon either focal distance is equal to the semi-focal axis.

12. Prove that if the normal at P to a central conic meet the focal axis at G and the non-focal axis at G', then $PG' : PG = CA^2 : CB^2$.

13. If the tangent and normal at P meet the non-focal axis at T' and G', prove that S, S', P, T', G' are concyclic and $CG' . CT' = CS^2$.

14. If from any point T on the tangent at P there be drawn perpendiculars TL and TN to SP and the directrix $SL : TN =$ eccentricity.

15. If CD be the diameter conjugate to CP, show that

$$SP . S'P = CD^2.$$

16. Show that points of contact of tangents from the foci to the auxiliary circle lie on the asymptotes.

17. The focus of a conic slides on a fixed line, the conic itself sliding on a fixed perpendicular line. Find the locus of the centre.

18. A rectangular piece of paper $ABCD$ is folded so that the corner C falls on the opposite side AB. Show that the crease envelops a parabola of which C is the focus and AB the directrix.

19. The vertex of a constant angle moves on a fixed straight line, while one of its sides passes through a fixed point S. Show that the other side envelops a parabola, of which S is a focus.

20. Show that the two tangents from a point to a conic are equally inclined to the focal distances; and conversely that if S, H be fixed points and through a point P lines PT, PU be drawn so that the angles SPH, TPU have common bisectors, a conic can be described with S, H as foci to touch PT and PU.

21. A focus and three tangents to a conic are given. Construct the axes of the conic in position and length.

22. Find the locus of the focus of a parabola passing through two fixed points A, B and the direction of whose axis is given.

23. Prove that a line-pair may be looked upon as the limiting case of a hyperbola when the foci coincide with the centre.

24. Prove that a point-pair may be looked upon as the limiting case of a very flat ellipse or hyperbola, the foci being coincident with the vertices. Show that the eccentricity of a point-pair is unity.

25. Prove that the pole of the tangent at P to a central conic with regard to the auxiliary circle lies on the ordinate of P.

26. If SY, SZ be perpendiculars from a focus S to tangents TP, TQ, the perpendicular from T to YZ passes through the other focus S'.

27. Find the locus of the vertices of the conjugate parallelograms of an ellipse.

28. If PP', DD' be conjugate diameters of a hyperbola and Q any point on the curve, show that $QP^2+QP'^2$ differs from $QD^2+QD'^2$ by a constant quantity.

29. If TP, TQ be tangents from a point T to a central conic, S, H the foci, show that the bisectors of the angle PTQ meet the non-focal axis in two fixed points when T describes the circle STH.

30. From a point P on a hyperbola PN is drawn perpendicular to the transverse axis and from N a line is drawn to touch the auxiliary circle at T. Prove that $TN:PN=$ ratio of semi-transverse to semi-conjugate axis.

31. Show that two parabolas which have a common focus and their axes in opposite directions intersect at right angles.

32. PQR being a triangle circumscribed to a parabola, prove that the perpendiculars from PQR to SP, SQ, SR are concurrent.

[Use the result of Art. 106 that the circumcircle of such a triangle passes through the focus.]

33. If a chord QQ' of a parabola meet a diameter PV at O, and if QV, $Q'V'$ be ordinates to this diameter, prove that $PV.PV'=PO^2$.

34. If from a point T outside a parabola a tangent TP and a chord TQQ' be drawn, and if the diameter through P meet QQ' at K, show that $TQ.TQ'=TK^2$.

35. Show that the chord of curvature to a parabola at P, drawn parallel to the axis, is $4SP$.

36. If the normal at P to a parabola meet the directrix at H, then the radius of curvature at $P=2.HP$.

37. Show that any point of a rectangular hyperbola is a point of trisection of the intercept of the normal at the point between the centre of curvature and the point where the normal meets the curve again.

38. Prove that through any point P of a conic, three circles of curvature of the conic pass.

[Let PQ, PQ' be chords equally inclined to axes, CR the diameter conjugate to PQ' meeting PQ at S. $P[Q] \,\overline{\wedge}\, P[Q']$ (oppositely equal); $P[Q'] \,\overline{\wedge}\, C[R]$ (conjugate); $\therefore P[Q] \,\overline{\wedge}\, C[R]$. The three points other than P where the conic locus of S meets the original conic have their circles of curvature passing through P, for at such points Q, R, S coincide and the tangent at Q is equally inclined to the axes with PQ.]

39. Show that the central chord of curvature of a conic at $P=2.CD^2/CP$, CD being the semi-diameter conjugate to CP.

EXAMPLES VI B.

1. Show how to cut the hyperbola whose equation in Cartesian coordinates is $\frac{x^2}{4} - \frac{y^2}{9} = 1$ from a right circular cone.

2. A right circular cone of semi-vertical angle 60° is cut by a plane inclined at an angle of 15° to the axis of the cone and whose perpendicular distance from the vertex is 4 inches. Construct the foci, vertices and asymptotes of the section.

3. A hyperbola has the points $(\pm 2, 0)$ for the extremities of its transverse axis and passes through the point (5, 7). Construct its asymptotes geometrically and find the real length of its conjugate axis.

4. Draw the hyperbola whose directrix is $x = 0$, focus (2, 0) and eccentricity 2·5.

5. Construct the directrices of the possible conics which have the origin for focus and pass through the three points $(-1, 1)$, (3, 3), (5, 0).

6. The focus of a conic is the origin and the conic passes through the point (3, 4). If the semi-latus rectum $= 3·5$ and the eccentricity $= \frac{1}{2}$, construct the second foci of the conics which satisfy the conditions.

7. The asymptotes of a hyperbola are parallel to the lines $x^2 - 4y^2 = 0$. One focus is the point (3, 0) and the semi-latus rectum $= 2$. Find the asymptotes in position and draw the curve.

8. A conic has the axis of y for directrix, the point (3, 0) for focus and eccentricity $= 2$. Draw a chord through the focus which shall be 3·5 units long.

9. A hyperbola has the lines $y = \pm 1·5x$ for asymptotes and passes through the point (5, 4). Construct its foci and vertices.

10. A conic touches the lines $y = x$, $3y - x = 3$, $2x + 3y = 12$ and has the point (2, 0) for focus. Construct its axes in position and length.

CHAPTER VII.

SELF-CORRESPONDING ELEMENTS.

110. Projective ranges and pencils of the second order. The points of a conic, like the points of a straight line, may be spoken of as forming a range, but such a range is said to be of the second order, the linear range being considered of the first order.

Similarly the tangents to a conic are said to form a pencil of the second order.

These are termed forms of the second order, and the conic to which they belong is called their base.

Ranges and pencils of the second order will be denoted by writing 2 as an index outside the bracket enclosing the typical element: thus $[P]^2$, $[p]^2$.

Two ranges of the second order are said to be projective (it will be shown in Chapter VIII that they can actually be projected into one another) if the pencils which they determine at any points of their respective bases are projective.

Similarly two pencils of the second order are said to be projective if the ranges which they determine on any tangents to their respective bases are projective.

From the known properties of projective pencils and ranges of the first order given in Chapter II it follows that two corresponding triads entirely determine the relation between two projective forms of the second order. Also if we define the cross-ratio of four points on a conic as the cross-ratio of the pencil which they determine at any point of the conic and the cross-ratio of four tangents to a conic as the cross-ratio of the range which they determine on any tangent to the conic, then projective forms of the second order are equi-anharmonic and conversely.

Again, as in Chapter II, two cobasal forms of the second

order cannot have more than two self-corresponding elements without being entirely coincident.

111. Cross-axis and cross-centre of cobasal projective forms of second order. Let P, P' (Fig. 41) be two corresponding points of two projective ranges $[P]^2$, $[P']^2$ lying on the same conic s.

Let A, A' be any given corresponding points of these ranges. Project the range $[P']^2$ from A as vertex, $[P]^2$ from A' as vertex. The pencils $A[P']$ and $A'[P]$ are projective and they have a self-corresponding ray $A'A$. Hence they are perspective and rays AP', $A'P$ meet at U on a fixed axis x.

This axis x is independent of the choice of the points A, A'.

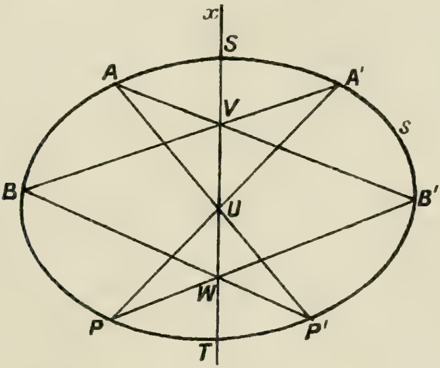


Fig. 41.

For let B, B' be any other pair of corresponding points. Then by the previous result AB', $A'B$ meet at V on x. Now consider the Pascal hexagon $AB'PA'BP'$. We have $(A'B, AB')$ $(AP', A'P)$ $(PB', P'B)$ are collinear. x is therefore the Pascal line and PB', $P'B$ meet at W on x. The same line x is therefore reached if we start from A and A', or if we start from B and B'.

There is thus a fixed line x on which meet the cross-joins AB', $A'B$ of any two corresponding pairs. This we shall call, as in the case of linear ranges, the *cross-axis*.

By reciprocation, or by proceeding in a manner similar to the above and using Brianchon's theorem, we reach the result that two projective pencils of tangents to the same conic have a *cross-*

centre, through which pass the joins of cross-meets $(ab', a'b)$ of corresponding pairs.

112. Self-corresponding elements of cobasal projective forms of second order. As in the case of forms of the first order, two cobasal projective forms of the second order cannot have more than two self-corresponding elements; for if they have three, say A, B, C and if P, P' be any other pair of corresponding elements, $\{ABCP\} = \{ABCP'\}$ and as in Art. 25 $P' = P$.

These self-corresponding elements may be constructed as follows. If the cross-axis of two projective ranges of the second order $[P]^2$, $[P']^2$ lying on the same conic s meet s at points S, T (Fig. 41) the points S, T are self-corresponding points of the ranges $[P]^2$, $[P']^2$.

For by the property of the cross-axis AT, $A'T$ meet s at a pair of corresponding points. But they both meet s at T. Hence a pair of corresponding points coincide at T, or T is self-corresponding. Similarly S is self-corresponding.

If the cross-axis x is itself a tangent to s, the self-corresponding points S, T coincide. If x do not meet the conic at real points, there are no real self-corresponding points.

Reciprocating, we have the theorem: the self-corresponding lines of two projective pencils of the second order belonging to the same conic are the tangents from the cross-centre. There are two real self-corresponding lines if the cross-centre is outside the conic: these coincide if the cross-centre is on the conic. If the cross-centre be inside the conic there are no real self-corresponding lines.

113. Two corresponding elements of two cobasal projective forms determine with the self-corresponding elements a constant cross-ratio. It will be sufficient to prove this for two projective ranges on the same conic, since all other cases can clearly be made to depend upon this. Now from Fig. 41, if AP', $A'P$ be two chords meeting on ST, then P, P' are corresponding points of the ranges determined by the triads S, T, A; S, T, A'. On the other hand it is obvious from symmetry that A', P' are corresponding points in the ranges determined by the triads S, T, A; S, T, P. Hence the cross-ratio of the four points $STAA'$ is equal to the cross-ratio of the four points $STPP'$, which proves the theorem.

114. Construction of self-corresponding points. The results of Art. 112 provide us with a construction for deter-

mining the self-corresponding elements of two cobasal projective forms of the first order.

Thus let there be two projective pencils having a common vertex O: let $a_1b_1c_1$; $a_2b_2c_2$ be two corresponding triads.

Describe any conic (in practice a circle will be a convenient conic to use) passing through O, and meeting a_1, b_1, c_1, a_2, b_2, c_2 at A_1, B_1, C_1, A_2, B_2, C_2 respectively. Construct the cross-axis of the ranges of the second order defined by $A_1B_1C_1$, $A_2B_2C_2$. This cross-axis is obtained from any two pairs of cross-joins (A_1B_2, A_2B_1) and (A_1C_2, A_2C_1).

The points S, T where this cross-axis meets the conic are self-corresponding points of the ranges of second order. The rays OS, OT are then self-corresponding rays of the given pencils of first order, since corresponding rays of these pencils pass through corresponding points of the ranges of second order.

On the other hand let there be two projective ranges on the same straight line u, defined by corresponding triads $A_1B_1C_1$, $A_2B_2C_2$.

Describe any conic (here again in practice a circle) touching u. From A_1, B_1, C_1, A_2, B_2, C_2 draw tangents a_1, b_1, c_1, a_2, b_2, c_2 to this conic. Construct the meets of (a_1b_2, a_2b_1) and (a_1c_2, a_2c_1). This is the cross-centre. The two tangents from the cross-centre meet u at the self-corresponding points of the given ranges.

Otherwise thus: the two ranges may be projected from any vertex and the self-corresponding rays of the concentric projective pencils so formed may be found by the construction given at the beginning of this article. They meet u at the self-corresponding points of the ranges.

115. Intersections of a straight line with a conic given by five points. Let O, O', A, B, C be the five points on the conic, u any straight line.

The conic is the product of the two pencils defined by $O(ABC)$, $O'(ABC)$.

If OA, OB, OC meet u at A_1, B_1, C_1 and $O'A$, $O'B$, $O'C$ meet u at $A_2B_2C_2$, the pencils $O(ABC)$, $O'(ABC)$ determine upon u two projective ranges of which $A_1B_1C_1$, $A_2B_2C_2$ are corresponding triads.

Find the self-corresponding points of these ranges on u by either of the methods given in the last article. Let these be S, T. Then OS, $O'S$ are corresponding rays of the pencils $O(ABC)$, $O'(ABC)$.

Therefore S is a point on the conic.

Similarly T is a point on the conic.

Hence S, T are the intersections of u with the conic.

116. Directions of asymptotes of a conic given by five points. If in the construction of the preceding Article the line u be the line at infinity $A_1B_1C_1$, $A_2B_2C_2$ are at infinity. Projecting $A_2^\infty B_2^\infty C_2^\infty$ from O we have two projective pencils having O for a common vertex. The rays OA_2^∞, OB_2^∞, OC_2^∞ are parallel to $O'A_2^\infty$, $O'B_2^\infty$, $O'C_2^\infty$, i.e. to $O'A$, $O'B$, $O'C$. Thus the pencil $O(A_2^\infty B_2^\infty C_2^\infty)$ is obtained by drawing through O parallels to the rays of $O'(ABC)$. Find the self-corresponding rays of the pencil $O(A_2^\infty B_2^\infty C_2^\infty)$ and the pencil $O(ABC)$; these rays lead to the points T^∞, S^∞ where the line at infinity meets the conic. They give therefore the directions

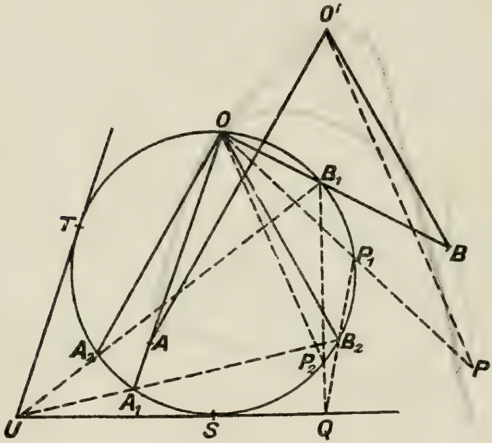


Fig. 42.

of the asymptotes. The asymptotes are then constructed in position by the method of Art. 76.

117. Construction of the parabolas through four given points. Let O, O', A, B be the four given points (Fig. 42). Through O draw any circle meeting OA, OB at A_1B_1 and the parallels through O to $O'A$, $O'B$ at A_2B_2.

Then if P is any point on the parabola and P_1, P_2 are the points where OP and the parallel through O to $O'P$ meet the circle, $[P_1]^2$, $[P_2]^2$ are two projective ranges on the circle whose self-corresponding points are the points corresponding to

the points at infinity on the curve, since when P is at infinity OP, $O'P$ are parallel.

In the case of the parabola the points at infinity are coincident because the line at infinity touches the curve. Hence the self-corresponding points of the ranges $[P_1]^2$, $[P_2^2]$ are coincident or the cross-axis touches the circle (Art. 112).

But we know one point on the cross-axis, namely the point U where A_1B_2 meets A_2B_1. The cross-axis is therefore either of the two tangents from U to the circle. The join of O and the point of contact of the cross-axis with the circle give the direction of the point at infinity on the parabola, or the direction of the axis. Having the direction of the axis and four points on the curve we may construct the parabola by the method of Art. 79, or more directly as follows. Take any point Q on the cross-axis. Join QB_2 meeting the circle at P_1, QB_1 meeting the circle at P_2. The parallel through O' to OP_2 meets OP_1 at a point P on the parabola.

Since two tangents can be drawn to a circle from U, the problem is in general capable of two solutions. These solutions are coincident if U be on the circle. In this case either A_1 and B_1 (or A_2 and B_2) coincide, that is, three of the given points are collinear and the conic then degenerates into two parallel straight lines, which is a special case of a parabola; or else A_1 and A_2 (or B_1 and B_2) coincide; A (or B) is then at infinity, so that three points and the direction of the axis are given and the parabola can be drawn by Pascal's theorem. If U be within the circle there are no real solutions to the problem.

118. Rectangular hyperbola through four points. Case of failure. The same principle will enable us to construct the rectangular hyperbola through four given points O, O', A, B. Draw a circle through O and find the points A_1, B_1, A_2, B_2 and the point U on the cross-axis by the same construction as before. Now since in the rectangular hyperbola the asymptotes are to be at right angles the self-corresponding rays of the pencils $O[P_1]$, $O[P_2]$ are at right angles, that is, they meet the circle at the extremities of a diameter; or the cross-axis of $[P_1]^2$, $[P_2]^2$ is a diameter. Hence join U to the centre of the circle, and we have the cross-axis required. The joins of O to its intersections with the circle give the directions of the asymptotes. Having these and four points on the curve we can construct the curve by Art. 76 or directly from the present construction as explained in the last article.

If U be at the centre of the circle, any diameter may be

taken as the cross-axis and an infinite number of rectangular hyperbolas may be drawn through the four points. In this case A_2B_1, A_1B_2 being diameters, OA, OB are perpendicular to OB_2, OA_2, that is, to $O'B$, $O'A$, or O is the orthocentre of the triangle $O'AB$. It is easy to prove that when this is so any one of the four given points is the orthocentre of the triangle formed by the other three.

119. Tangents from any point to a conic given by five tangents. Let t, t', a, b, c be five tangents to a conic.

Let A, B, C be the points where t meets a, b, c,

A', B', C' " " " t' " a, b, c.

Let O be any point in the plane.

If p be any tangent to the conic meeting t at P and t' at P':

The ranges $[P]$, $[P']$ are projective: hence the pencils $O[P]$, $O[P']$ are projective.

If p passes through O, OP and OP' are coincident.

Therefore the tangents to the conic through O are the self-corresponding rays of the pencils $O[P]$, $O[P']$.

Determine these self-corresponding rays from the triads $O(ABC)$, $O(A'B'C')$ by the method of Art. 114 and these give the tangents required.

EXAMPLES VII A.

1. A conic passes through five points O, O', A, B, C. Show how to construct graphically its intersections with any circle through O, O' without drawing the conic. Prove that the common chord not passing through OO' is always real, even when the circle does not meet the conic again in real points.

2. A conic is given by five points. Without drawing the curve find a test to determine whether it is an ellipse, hyperbola or parabola.

3. Prove that two conics can be drawn through four given points such that their asymptotes make an angle α with one another: and show how to construct them.

[In the construction of Art. 117 the cross-axis must cut off a constant arc from the circle through O and therefore touches a circle concentric with this circle.]

4. Investigate the nature of the simple quadrilateral formed by four points if it is impossible to draw a real parabola through them.

5. Directly equal ranges on a circle may be defined as ranges in which two directly equal pencils whose vertices are on the circle meet the circle. Show that the cross-axis of two such ranges is at infinity.

6. Oppositely equal ranges on a circle may be defined as ranges in which two oppositely equal pencils whose vertices are on the circle meet the circle. Show that the cross-axis of two such ranges passes through the centre.

7. Given five points on a conic draw the tangents to it from any point in the plane.

[Find where two rays through the point cut the conic. Hence construct the polar.]

8. A_1B_1, A_2B_2 are two corresponding pairs of points of two collinear ranges. Given that the self-corresponding points of the two ranges are coincident, find the point at which they coincide.

EXAMPLES VII B.

1. A, B, C, D are four points on a straight line at unit distance apart in order. ABC, DCA define two projective ranges. Construct the self-corresponding points of these ranges.

2. Draw an indefinitely long line Ox, and on it take A, B, C such that $AB=3$, $BC=2$. Take also on Ox three points A', B', C' such that $CC'=6$, $CB'=10$, $CA'=12$. It is required to find the position of a point F on Ox such that the cross-ratios $\{ABCF\}$ and $\{A'B'C'F\}$ shall be the same. Verify your construction by algebraic calculation.

3. O, O' are two points 4″ apart: through O are drawn three rays OA, OB, OC making with OO' angles of 90°, 60°, 30° (counter-clockwise); and through O' are drawn three rays $O'A$, $O'B$, $O'C$ making with $O'O$ angles of 30°, 15°, 75° (clockwise).

Without drawing the curve construct the asymptotes of the locus of intersections of corresponding rays of the pencil defined by the triads $O(ABC)$, $O'(ABC)$.

4. Find the directions of the axes of the parabolas which can be drawn through the four points whose rectangular coordinates are

$$(-\cdot 5, -1\cdot 5), (4, 0), (-\cdot 9, -\cdot 4), (7\cdot 5, -1\cdot 5).$$

5. Construct the rectangular hyperbola through the four points (0, 0), (0, 2), (1, 0), (1, 3), the axes being rectangular.

6. The angle between the axes of x, y being 45° a conic touches the lines $2x+y=2$, $3x+10y=30$, $x+y=5$ and the axes. Without drawing the curve, construct the two tangents to it from the point $(4, -3)$.

7. The following points are given: $O(0, 0)$, $O'(3, 0)$, $A(-1, 4)$, $B(2, 2)$, $C(6, 5)$, the axes being rectangular. $O(ABC)$, $O'(ABC)$ define two projective pencils. Construct the rays of the first pencil which are parallel to the corresponding rays of the second pencil.

CHAPTER VIII.

IMAGINARIES AND HOMOGRAPHY.

120. Point and line coordinates in a plane. The position of a point P in a plane may be defined by two coordinates x, y given by the intercepts cut off, on two fixed axes, between their intersection or origin and the parallels through P to the axes. In this system of coordinates the coordinates of the points of any straight line satisfy an equation of the first degree

$$Ax + By + C = 0.$$

If we divide this equation by C it takes the form

$$lx + my + 1 = 0.$$

A straight line is therefore completely defined when we know the two coefficients l, m. These may then be spoken of as the coordinates of the line.

The coordinates of the points of a curve satisfy a relation which is called the Cartesian equation of the curve.

In like manner the coordinates of the tangents to a curve satisfy a relation which is called the tangential equation of the curve.

If the coordinates l, m of a line satisfy a relation of the first degree, this can be put into the form

$$la + mb + 1 = 0,$$

and this shows that the line whose coordinates are l, m passes through the point whose coordinates are a, b.

An equation of the first degree in l, m is therefore the tangential equation of a point and the lines whose coordinates satisfy this equation are rays of a pencil.

If $l = 0$, $m = 0$, x or y or both must be infinite if $lx + my$ is to be equal to the finite quantity -1. Hence $l = 0$, $m = 0$ are the coordinates of the line at infinity. Similarly if $x = 0$, $y = 0$ the lines through the origin must have l or m or both infinite.

Notice the duality implied by this arrangement of point and line coordinates. By giving the symbols a different interpretation and taking l, m as coordinates of a point, x, y as coordinates of a line and bearing in mind the symmetry of the relation of incidence $lx + my + 1 = 0$ in x, y and l, m respectively, we see that to any geometrical theorem corresponds another in which points and lines are interchanged. This is the principle of duality which we have already deduced from the theory of reciprocal polars in Art. 65. The present result shows that this principle is entirely independent of the theory of reciprocal polars.

121. Point and plane coordinates in space. In like manner the position of a point P in space may be defined by taking three axes OX, OY, OZ through an origin O and drawing through O planes parallel to YOZ, ZOX, XOY to meet OX, OY, OZ respectively at L, M, N. Then the segments OL, OM, ON taken with proper sign are denoted by x, y, z and called the coordinates of the point. It is shown in treatises on analytical geometry (see Salmon, *Geometry of Three Dimensions*, or C. Smith, *Solid Geometry*) that in this system of coordinates a plane is represented by an equation of the first degree in the coordinates which may be put into the form

$$lx + my + nz + 1 = 0 \quad\ldots\ldots\ldots\ldots\ldots\ldots (1),$$

and conversely that every such equation defines a plane.

(l, m, n) may be called the coordinates of the plane and the above equation expresses that the plane (l, m, n) and the point (x, y, z) are incident.

The coordinates of a point on a surface satisfy a single equation in x, y, z which is called the Cartesian equation of the surface.

The coordinates of a plane tangent to a surface satisfy a single equation in l, m, n which is called the tangential equation of the surface.

The equation (1) expresses, when x, y, z are treated as constants and l, m, n as variables, that the coordinates of the planes passing through x, y, z satisfy the equation (1) of the first degree in l, m, n.

Conversely such an equation of the first degree in l, m, n represents a set of planes through a point. Such a set of planes is called a *sheaf* of planes and the point through which they pass is called the vertex of the sheaf.

An equation of the first degree in l, m, n is therefore the tangential equation of a point.

As in Art. 120, $l=0$, $m=0$, $n=0$ are the coordinates of the plane at infinity, whereas $x=0$, $y=0$, $z=0$ correspond to infinite plane-coordinates.

122. Principle of Duality in space. The symmetrical form of the equation

$$lx+my+nz+1=0$$

implies that if the point (x, y, z) and the plane (l, m, n) are incident, so are the plane (x, y, z) and the point (l, m, n). Thus to any theorem connecting points and planes, there corresponds a reciprocal theorem connecting planes and points, obtained from the first by interchanging the interpretations of x, y, z and l, m, n. In this translation the join of two points corresponds to the meet of two planes. Hence a straight line corresponds to a straight line. To the set of lines through a point, which is called a *sheaf of lines*, corresponds the set of lines in a plane, which is called a *plane of lines.* To a sheaf of planes through a point corresponds the set of points of a plane, which is called a *plane of points.* To a range of points on a line corresponds a set of planes through a line or *axis*, which is called an *Axial Pencil.* To a set of lines through a point and lying in a plane (a flat pencil) corresponds a set of lines lying in a plane and passing through a point (another flat pencil). To a point on a surface corresponds a tangent plane to the corresponding surface. To the tangent plane at a point corresponds the point of contact of a tangent plane.

To the points where a straight line cuts a surface correspond the tangent planes drawn through a line to the corresponding surface.

The degree of a surface being defined as the number of points in which it is cut by any line and the class of a surface as the number of tangent planes which can be drawn to it through any line, it follows that the degree of a surface is equal to the class of its reciprocal surface.

123. Cross-ratio of an axial pencil. An axial pencil of four planes $\alpha, \beta, \gamma, \delta$ through a line x, has a definite cross-ratio. For cut it by any two straight lines u_1, u_2. These meet $\alpha\beta\gamma\delta$ in ranges $A_1B_1C_1D_1$, $A_2B_2C_2D_2$ respectively. On x take two points V_1, V_2. The planes u_1V_1, u_2V_2 meet in a line u_3 which cuts $\alpha\beta\gamma\delta$ in a range $A_3B_3C_3D_3$. Then the ranges $A_1B_1C_1D_1$, $A_3B_3C_3D_3$ are perspective from V_1; and the ranges $A_3B_3C_3D_3$, $A_2B_2C_2D_2$ are perspective from V_2. Hence we have

$$\{A_1B_1C_1D_1\}=\{A_3B_3C_3D_3\}$$
$$\{A_3B_3C_3D_3\}=\{A_2B_2C_2D_2\}.$$

That is, $\{A_1B_1C_1D_1\} = \{A_2B_2C_2D_2\}$.

Hence all straight lines meet an axial pencil of four planes in ranges having the same cross-ratio. This cross-ratio is defined to be the cross-ratio of the axial pencil.

An axial pencil, like a range and a flat pencil, is known as a one-dimensional geometric form of the first order.

124. Imaginary elements. We are now in a position to introduce into Geometry a new set of ideal elements, which are called imaginary elements. In Art. 4 elements at infinity were introduced, in order to enable us to state theorems on the straight line in all their generality, without having to consider cases of exception. Thus, after the introduction of the elements at infinity, we were able to state, quite generally, that coplanar lines always have a point of intersection, that a straight line and a plane always have a point of intersection, that two planes always have a straight line in common.

But, as we proceeded, we met another set of cases of exception which could not be dealt with in the same manner. For example two collinear projective ranges might have two real self-corresponding points, or they might have none. Nevertheless the nature of two such ranges is not intrinsically different in the two cases, as appears from the fact that any property which we prove for two such ranges which have self-corresponding points holds equally for ranges not having self-corresponding points, provided the property does not involve the self-corresponding points.

In like manner a straight line may cut a circle or conic in two points, or it may not cut the curve at all. Two tangents may be drawn from a point to a conic, or none may be drawn.

The validity of the results we have reached therefore depends on the elements of the figures having certain relative positions, without which some of the results apparently disappear.

Now it would be extremely convenient if these restrictions could be removed and if, by introducing a new set of ideal elements, which have no visual existence, we could state our theorems in a perfectly general manner.

Such ideal elements are provided for us by the method of coordinates explained in Arts. 120, 121.

For any geometrical theorem can be translated into an algebraic theorem connecting point and line coordinates (or point and plane coordinates). If in this theorem certain real elements appear, the coordinates of these elements can be deduced from

the solution of certain algebraic equations, involving the data. If by altering the numerical values of these data, without altering their nature, these elements disappear from the geometrical theorem, they will not disappear from the algebraic theorem, for an algebraic equation continues to have solutions, even when its constants are such that these solutions are not real. The algebraic solution will therefore still give values for the coordinates of those elements which have disappeared from the geometrical solution, but these coordinates will be complex, that is of the form $a + ib$, where $i = \sqrt{-1}$ and a, b are real. The points, straight lines or planes defined by such coordinates have no visual existence; nevertheless all analytical theorems remain true of them and therefore all geometrical operations, which are interpretable by means of analysis, will continue to hold for such imaginary elements. And this is true not only of points, straight lines and planes, but of all curves and surfaces of higher degree.

Thus the locus

$$x^2 + y^2 = -a^2$$

is not a real circle: nevertheless it possesses, analytically, all the properties of a circle and, if we admit imaginary elements, we may perform with it the operations which we can perform with an ordinary circle.

We will therefore, from this point onwards, assume the existence of such imaginary elements, so that if a construction which leads to certain elements in one case fails to lead geometrically to such elements in another case, we shall say that those elements are still there, but are imaginary.

Thus we know that two projective collinear ranges will generally have two self-corresponding points. This shows that the problem of determining the self-corresponding points of two such ranges is analytically capable of two solutions. Hence it will have two analytical solutions in *all* cases. We shall then say that two such ranges have *always* two self-corresponding points, but that these may be *real* or *imaginary*.

In the same way a straight line will be conceived as always cutting a conic at two points, real or imaginary; and from a point two tangents, real or imaginary, can always be drawn to a conic.

Again we know that, in general, two distinct conics will intersect in four points. The problem of finding the intersections of two conics has therefore four analytical solutions. We shall say that it has always four geometrical solutions, that

is, every two conics have four points of intersection, real or imaginary.

The student may object that this introduction of imaginary elements is really, from the geometrical point of view, a mere verbal delusion, for in what way can we derive help in practice from a construction in which one or more steps are imaginary? The answer is that these imaginary elements cannot, indeed, be used in drawing-board *constructions*, but it may, and does, happen that a *demonstration*, involving such imaginary elements, leads to a result which is free from them. Thus by means of imaginary points and lines we can obtain real theorems, precisely as we can, by means of points and lines at infinity, obtain theorems relating to figures at a finite distance.

It is true that in all cases such theorems might be obtained by reasoning with purely real elements. But such proofs are often exceedingly complicated; also two theorems which, when we use imaginary elements, are only particular cases of the same theorem, require, if we restrict ourselves to real elements, proofs which are not infrequently quite dissimilar. The simplicity and unity obtained by the introduction of imaginary elements add very greatly in power to the methods of geometry.

125. Conjugate Imaginaries. If the coordinates of an element are of the form $a+ib$, the element whose coordinates are obtained from those of the first by changing the sign of i is said to be a conjugate imaginary to the first element.

Thus the point $(0, -i, 1+i)$ is the conjugate imaginary point to $(0, i, 1-i)$.

If two elements are incident, their conjugate imaginary elements are also incident.

For any equation involving imaginaries may be reduced to the form $U+iV=0$, where U and V are real. We have therefore $U=0$, $V=0$, and therefore $U-iV=0$, that is, the equation obtained by changing the sign of i everywhere is also satisfied.

It follows similarly that if a real and an imaginary element are incident, the real element and the conjugate imaginary element are also incident. For a real element may be looked upon as its own conjugate imaginary.

If an element A of any nature is determined by two other elements P, Q (points, planes or intersecting lines), its conjugate imaginary element A' is determined by the conjugate imaginary elements P', Q'. For since A, P are incident $\therefore$ A', P' are incident; and since A, Q are incident $\therefore$ A', Q' are incident. Hence $A'=P'Q'$. In particular if $Q=P' \therefore Q'=P$ or $A'=P'P=A$.

Hence the element (if any) determined by two conjugate imaginary elements is always real.

In particular the join of two conjugate points or the meet of two conjugate planes are real lines. Two conjugate lines which intersect determine a real point of intersection and a real plane.

The elements determined by a real element A and two conjugate imaginary elements P, P' are conjugate imaginary. For A being its own conjugate imaginary, AP is the conjugate imaginary to AP'.

Also, if S be any locus or envelope which is real or into whose analytical equation only real coefficients enter, and P be any imaginary element incident with S (i.e. lying on or tangent to S), the relation of incidence is expressed by an equation

$$U + iV = 0.$$

This implies $$U - iV = 0.$$

But the latter is what we obtain if we change the sign of i in the coordinates of P, since the coefficients of the equation for S do not contain i. Hence P' is also incident with S.

It follows that if two real loci have one imaginary intersection P, the conjugate imaginary point P' is also an intersection, since it must lie on both curves. The corresponding chord PP', being determined by two conjugate elements, is real.

126. Number of real elements incident with an imaginary element. An imaginary point has only one real line through it, namely the one joining it to its conjugate imaginary point. For if it had two it would be the intersection of two real lines and therefore a real point.

Similarly an imaginary plane has only one real line lying in it, namely its intersection with its conjugate imaginary plane. For a plane through two real lines is a real plane.

An imaginary line, for the same reason, cannot have two real points on it. But imaginary lines may be of two kinds. A line of the first kind has one real point on it. A line of the second kind has no real point on it.

By the last Article the conjugate imaginary line p' to a line p of the first kind passes through the real point on p. p, p' therefore intersect and, being conjugate, determine a real plane.

Thus a line of the first kind has one real plane passing through it. It cannot have a second, for it would then be the meet of two real planes and so be a real line.

Conversely, if an imaginary line p has one real plane passing

through it, its conjugate imaginary line p' lies in this plane and meets p at a real point, so that p is of the first kind.

A line of the second kind has therefore no real plane through it, as well as no real point on it, and it does not intersect its conjugate imaginary line.

Such lines may be obtained by taking conjugate imaginary pairs P, P' and Q, Q' on non-intersecting real lines a, b respectively. Then P, P', Q, Q' cannot be coplanar and the lines PQ, $P'Q'$ are conjugate imaginary lines which do not intersect.

127. Homographic Ranges. Let x be the distance of a point P on a line u from a given origin O on the line. Let x' be the distance of a point P' on another line u' from an origin O' on that line.

Let a correspondence be established between the ranges of such a nature that to any point P (real or imaginary) corresponds one point P' (real or imaginary) and one only, and conversely to every point P' corresponds one point P and one only. And let the correspondence be algebraic, that is, let the relation between P and P' be expressible by means of a rational integral algebraic equation between x and x', that is, an equation in which only sums of positive powers or of products of positive powers of x and x' appear equated to zero. No transcendental functions such as $\sin x$, $\log x$, e^x, etc. are to appear in the relation between x and x'.

Since for a given value of x there is one value of x' and one only, the equation can involve only the first power of x'; and since for a given value of x' there is only one value of x, it can involve only the first power of x.

It will therefore take the form

$$Axx' + Bx + Cx' + D = 0 \quad \text{..................(1)}.$$

Two ranges between which such a one-one correspondence exists are said to be *homographic*.

Projective ranges are clearly homographic: for their correspondence is one-one and the relation between the coordinates of a point and of its projection on any plane is certainly algebraic and rational.

128. Homographic ranges are equi-anharmonic. The relation (1) of Art. 127 leads to

$$x' = -\frac{Bx + D}{Ax + C}.$$

Let x_1, x_2, x_3, x_4 be the x's of four points P_1, P_2, P_3, P_4 on

u; and x_1', x_2', x_3', x_4' the x''s of the four corresponding points P_1', P_2', P_3', P_4'.

Then

$$\{P_1'P_2'P_3'P_4'\} = \frac{P_1'P_2' \cdot P_3'P_4'}{P_1'P_4' \cdot P_3'P_2'} = \frac{(x_2'-x_1')(x_4'-x_3')}{(x_4'-x_1')(x_2'-x_3')}.$$

Now $$x_2' - x_1' = -\frac{Bx_2+D}{Ax_2+C} + \frac{Bx_1+D}{Ax_1+C}$$

$$= \frac{(AD-BC)(x_2-x_1)}{(Ax_2+C)(Ax_1+C)}.$$

Hence $$\frac{(x_2'-x_1')(x_4'-x_3')}{(x_4'-x_1')(x_2'-x_3')} = \frac{(x_2-x_1)(x_4-x_3)}{(x_4-x_1)(x_2-x_3)},$$

the other factors all cancelling. Therefore

$$\{P_1'P_2'P_3'P_4'\} = \{P_1P_2P_3P_4\},$$

or the cross-ratio of four points of a range is equal to the cross-ratio of the four corresponding points of a homographic range.

It follows from the above that homographic ranges are projective. For given two homographic ranges construct two projective ranges having two corresponding triads the same as in the two homographic ranges. Then since both the projective and the homographic relation are equi-anharmonic, to any fourth point of one range will correspond the same fourth point of the other, whether projectively or homographically. The two given homographic ranges are therefore projective ranges.

129. Homographic flat pencils. If the rays of two flat pencils are connected by a one-one correspondence such that if m be any parameter in terms of which the coordinates of any ray of one pencil can be expressed linearly (and, conversely, which is uniquely determined when this ray is given), and if m' be a similar parameter for the other pencil, then m and m' are related by a rational algebraic equation: then this equation will be of the form

$$Amm' + Bm + Cm' + D = 0,$$

and the two flat pencils are said to be homographic.

Usually m, m' are the tangents of the angles made by the rays with fixed lines in the planes of the pencils.

It is clear that the ranges in which two such pencils will cut any transversals u, u' are likewise homographic. For the distances x, x' of the points of section measured along u, u' are connected with m, m' (and therefore with each other) by rational

algebraic relations; and also the correspondence between x, x' is seen to be one-one.

Since these homographic ranges are equi-anharmonic and projective, the two homographic pencils which stand on these ranges are also equi-anharmonic and projective.

Conversely projective pencils are homographic, since the correspondence between the rays is one-one and the relation between the coordinates of corresponding rays must clearly be both algebraic and rational.

130. Homographic axial pencils. In like manner two axial pencils whose planes correspond uniquely while the coordinates of corresponding planes are connected by an algebraic relation are said to be homographic.

The flat pencils in which two homographic axial pencils meet any two given planes are themselves homographic and projective.

The ranges in which two homographic axial pencils are met by any two given straight lines are homographic and projective.

Note that we cannot use the term projective of homographic axial pencils, since these are not plane forms and cannot therefore be projected into one another.

Two homographic axial pencils are entirely determined by two corresponding triads. For take two straight lines meeting the axial pencils in projective ranges, two corresponding triads of planes of the axial pencils determine on the lines two corresponding triads of points of the ranges. These determine the relation between the ranges and therefore the relation between the axial pencils.

Notice that if two homographic axial pencils have a common axis they have two self-corresponding planes, which correspond to the two self-corresponding points of the projective ranges in which the axial pencils are cut by any straight line.

131. Homographic unlike forms. If there be a one-one algebraic correspondence between the rays of a flat pencil and the points of a range, the two forms will still be spoken of as homographic.

Similarly a range and an axial pencil, or an axial pencil and a flat pencil, may be homographic.

Clearly from two unlike homographic forms may be derived, by projection or section, two like homographic forms.

A particular case of homographic unlike forms is furnished by the principle of duality, the correspondence between any element and its reciprocal being obviously one-one and algebraic.

132. Homographic ranges and pencils of the second order. If there be a one-one correspondence between the elements of two forms of the second order (ranges or pencils) which is expressible by an algebraic relation between the coordinates of the elements the forms are said to be homographic. A form of the second order may also be homographic with a form of the first order.

It is easy to show that if the forms of the second order are both ranges, or both pencils, such homographic forms are projective forms of the second order as defined in Art. 110.

For example, if we join two homographic ranges $[P_1]^2$, $[P_2]^2$ to vertices O, S lying on their respective bases, we obtain two pencils related by a one-one algebraic correspondence. These pencils are accordingly homographic and projective and the ranges $[P_1]^2$, $[P_2]^2$ are projective.

133. Geometrical evidence of Homography. It may be asked: when may we assert, from purely geometrical evidence, that the correspondence between two forms is homographic? For if we had to have recourse to analysis every time in order to apply the test whether the connecting relation is of the homographic type, the labour of calculation would in many cases be considerable, and the principle would be of little value in pure geometry.

We shall therefore suppose that our attention is to be confined to what are called algebraic curves or surfaces, that is, curves or surfaces whose equations are rational and integral in the coordinates. The conditions (a) that a point shall lie on such a locus, (b) that a straight line or plane shall touch such an envelope, are rational integral algebraic in the coordinates of the point, line, or plane. Therefore if a correspondence be established by means of the following processes: (1) taking joins of points or meets of planes, or planes through points and lines or meets of planes and lines; (2) finding intersections of algebraic curves or surfaces with straight lines or with other algebraic curves or surfaces; (3) drawing tangent lines or planes to such algebraic curves or surfaces, or finding points of contact of such tangent lines or planes (note that this includes finding common tangents to two curves or surfaces and also constructing polars): at each step, *provided we nowhere introduce an arbitrary restriction on our choice of alternatives*, an algebraic condition is brought in, which is rational and integral. In the process of elimination no radicals and no transcendental functions can be introduced (for the complete eliminant of two algebraic

equations for any variable is known to be a rational integral function of their coefficients). Hence the final relation between the coordinates is algebraic and rational.

The above justifies the statements made in Arts. 127, 129 that the coordinates of two corresponding points or lines of two projective forms are connected by rational algebraic relations. For clearly the processes of projection fall under the above headings.

We may note in passing that the same type of reasoning will show that any curve obtained from an algebraic curve by processes of this kind is likewise an algebraic curve. Thus the projection of an algebraic curve is an algebraic curve. In particular, the circle being an algebraic curve (its equation referred to rectangular axes through its centre being $x^2 + y^2 = r^2$), the conic is also an algebraic curve.

Next, as to being certain from geometrical evidence that the correspondence is really one-one. It should be borne in mind that the correspondence must be *intrinsically*, and not *accidentally*, one-one, that is, the fact of its being one-one must depend on the *intrinsic* nature of the curves used, such as their *degree* or *class*, and not on *accidental* characteristics, such as their *position* or *shape*. In this way alone can we be sure that the correspondence is still one-one when imaginary elements are taken into account, and without such assurance we cannot be sure that we are dealing with a homography.

For example the relation

$$x = x'^3$$

is a one-one relation between x, x' so far as real values are concerned, but it is not a homographic relation.

We may describe it geometrically thus :

Take a point P on the axis Ox whose coordinate is x. Draw through P a parallel to Oy meeting the straight line $y = x$ at P_1. Through P_1 draw a parallel to Ox meeting the cubic curve $y = x^3$ at P_2. Through P_2 draw a parallel to Oy meeting Ox at P'. P' is the point corresponding to P.

Put in this form the reason why the correspondence is not homographic is geometrically obvious. For although P_1P_2 meets $y = x^3$ in only one *real* point, the curve being of the third degree must be met by any straight line in *three* points. Thus there will be three points P_2 and therefore three points P' corresponding to one point P, but two of these are imaginary.

Again, if a line AB of constant length moves with its extremities on a fixed conic the pencils $O[A]$, $O[B]$, where O

is a fixed point on the conic, are not homographic. For the given condition is equivalent geometrically to stating that B is the intersection with the conic of a circle of fixed radius and centre A. This circle has four intersections with the conic, any one of which may be taken for B. Therefore to one ray OA should correspond four rays OB, and it is only by an *arbitrary convention* (to secure continuity of sliding motion) that this number is reduced to unity.

Note that this does not hold if AB slides on a fixed *circle s*. For then we may restate the problem as follows, since AB subtends a fixed angle at the circumference. Take a second fixed circle s' equal to the first. In it place a fixed chord ED equal to AB. Given any position of OA, draw EQ parallel to OA to meet s' at Q and OB is then parallel to QD. The correspondence is now clearly one-one.

The above will give the reader some notion of the limits within which the application of the principle of one-one correspondence is valid, but rapidity and certainty in recognizing these geometrically will be best ensured by the consideration of examples.

134. Every curve of the second degree is a conic. For let O, O' be two points on a curve of the second degree. Draw any ray OP through O: it meets the curve at one other point P, since O is already on the curve. Join $O'P$. Then if we start from OP, $O'P$ is uniquely determined. Conversely if we start from $O'P$, since O' is already on the curve, $O'P$ meets the curve again at one point only, hence OP is uniquely determined. $O[P]$, $O'[P]$ are therefore homographic pencils. Hence they are projective. Therefore by Art. 41 the locus of P is a conic.

In like manner we can show that every plane curve of the second class is a conic. For let t, t' be two tangents to the curve. On t take any point T. Through T one tangent p can be drawn to the curve and one only, meeting t' at T'. T, T' are seen to correspond uniquely. Hence $[T]$, $[T']$ are homographic and therefore projective: by Art. 42, TT' envelops a conic.

135. Notation for homography. The notation $\overline{\wedge}$ which was introduced in Art. 35 for "is projective with" will now be extended to homographic forms and be read "is homographic with." This notation will not contradict the previous, since projective forms are homographic.

136. Homographic plane figures. Let a one-one algebraic correspondence be established between the points of a plane figure ϕ and the points of another plane figure ϕ'. If in addition the transformation be such that to a straight line of ϕ corresponds a straight line of ϕ' and conversely, the two plane figures are said to be directly homographic or, more simply, homographic.

The relation between such figures will be called a *homography*.

Let x, y be the coordinates in the plane of ϕ of a point P of ϕ.

Let x', y' be the coordinates in the plane of ϕ' of the corresponding point P' of ϕ'.

Then if the correspondence between x, y and x', y' is to be one-one, x', y', when solved for, must not involve radicals containing x, y, that is, they must be rational functions of x, y. Reducing them to the same denominator we have

$$x' = \frac{P}{R}, \quad y' = \frac{Q}{R} \quad \text{.........................(1)},$$

where P, Q, R are polynomials in x, y.

To the straight line

$$l'x' + m'y' + 1 = 0$$

of the figure ϕ corresponds the locus

$$l'\frac{P}{R} + m'\frac{Q}{R} + 1 = 0 \quad \text{.......................(2)}$$

of the figure ϕ.

This locus (2) is not a straight line unless P, Q, R either reduce to expressions of the first degree in x, y or else have a common factor, such that when it is divided out of P, Q, R, the remaining factor is of the first degree.

In either case equations (1) reduce to the form

$$x' = \frac{A_1x + B_1y + C_1}{A_3x + B_3y + C_3}, \qquad y' = \frac{A_2x + B_2y + C_2}{A_3x + B_3y + C_3} \quad \text{......(3)},$$

and then the locus (2) becomes the straight line

$$l'(A_1x + B_1y + C_1) + m'(A_2x + B_2y + C_2) + A_3x + B_3y + C_3 = 0,$$

which being reduced to the form

$$lx + my + 1 = 0,$$

gives

$$l = \frac{A_1l' + A_2m' + A_3}{C_1l' + C_2m' + C_3}, \qquad m = \frac{B_1l' + B_2m' + B_3}{C_1l' + C_2m' + C_3} \quad \text{......(4)},$$

showing that the line coordinates transform according to a similar law.

The equations (3) can be written

$$(A_3x' - A_1)\,x + (B_3x' - B_1)\,y + (C_3x' - C_1) = 0,$$
$$(A_3y' - A_2)\,x + (B_3y' - B_2)\,y + (C_3y' - C_2) = 0.$$

Solving these for x we find

$$\left.\begin{aligned} x &= \frac{(C_3y' - C_2)(B_3x' - B_1) - (C_3x' - C_1)(B_3y' - B_2)}{(A_3x' - A_1)(B_3y' - B_2) - (A_3y' - A_2)(B_3x' - B_1)} \\ &= \frac{(B_2C_3 - B_3C_2)\,x' + (B_3C_1 - B_1C_3)\,y' + (B_1C_2 - B_2C_1)}{(A_2B_3 - A_3B_2)\,x' + (A_3B_1 - A_1B_3)\,y' + (A_1B_2 - A_2B_1)} \\ &\text{and similarly} \\ y &= \frac{(C_2A_3 - C_3A_2)\,x' + (C_3A_1 - C_1A_3)\,y' + (C_1A_2 - C_2A_1)}{(A_2B_3 - A_3B_2)\,x' + (A_3B_1 - A_1B_3)\,y' + (A_1B_2 - A_2B_1)} \end{aligned}\right\} \quad \text{.........(5).}$$

Equations (5) show that the transformation from xy to $x'y'$ is of the same type as the transformation from $x'y'$ to xy. We deduce that to a straight line of ϕ corresponds a straight line of ϕ', and one only, which can be otherwise established by solving back equations (4) for l', m'.

It is clear from the definition that corresponding ranges and corresponding pencils in two homographic figures are themselves homographic.

137. A plane homography is determined by two corresponding tetrads. Let $A_1B_1C_1D_1$, $A_2B_2C_2D_2$ be two *tetrads* or sets of four points in the plane figures ϕ_1, ϕ_2. These tetrads may be arbitrarily given, with the one restriction that no three points in either tetrad are to be collinear. Then a homographic correspondence can be established between ϕ_1 and ϕ_2 as follows.

Let P_1 be any point of ϕ_1. Draw through A_2 a ray A_2P_2 such that

$$A_2\{B_2C_2D_2P_2\} = A_1\{B_1C_1D_1P_1\} \quad \text{...........(1).}$$

There is only one such ray by Art. 25.

Also draw through B_2 a ray B_2P_2 such that

$$B_2\{A_2C_2D_2P_2\} = B_1\{A_1C_1D_1P_1\} \quad \text{...........(2).}$$

P_2, being the intersection of A_2P_2, B_2P_2, is determined uniquely when P_1 is given, and conversely. This construction then establishes between ϕ_1 and ϕ_2 a one-one point to point correspondence, which is easily verified to be algebraic.

To prove that it is a homography we have to show that if P_1 describes a straight line, P_2 describes another straight line.

Now from (1) and (2) above

$$A_2[P_2] \barwedge A_1[P_1],$$

A_2B_2 corresponding to A_1B_1, and

$$B_2[P_2] \barwedge B_1[P_1],$$

B_2A_2 corresponding to B_1A_1.

Now if P_1 describes a straight line, $A_1[P_1]$, $B_1[P_1]$ are perspective, A_1B_1 being self-corresponding. Hence

$$A_2[P_2] \barwedge B_2[P_2]$$

and A_2B_2 is self-corresponding: that is, $A_2[P_2]$, $B_2[P_2]$ are perspective and P_2 describes a straight line. The given construction therefore determines a homography.

Also it is the only homography in which A_1, B_1, C_1, D_1 correspond to A_2, B_2, C_2, D_2 respectively. For if P_1, P_2' be corresponding points in any other homography satisfying the given conditions, P_1, P_2' must satisfy the relations

$$A_2\{B_2C_2D_2P_2'\} = A_1\{B_1C_1D_1P_1\} \text{ ....(3)},$$

$$B_2\{A_2C_2D_2P_2'\} = B_1\{A_1C_1D_1P_1\} \text{ ....(4)},$$

since in a homography corresponding pencils are projective. Comparing (3) and (4) with (1) and (2) we see that $P_2' = P_2$.

The construction cannot fail unless two of the rays of one of the pencils $A_1(B_1C_1D_1)$, $A_2(B_2C_2D_2)$, $B_1(A_1C_1D_1)$, $B_2(A_2C_2D_2)$ coincide, that is, unless three of the points of either tetrad are in one straight line. In this case no homography can exist unless the three corresponding points are also in a straight line. But then the homography is no longer completely determined. For if A_1, B_1, C_1 be points on a straight line p_1 and A_2, B_2, C_2 the corresponding points on a straight line p_2, the triads $A_1B_1C_1$, $A_2B_2C_2$ determine completely the corresponding points of p_1, p_2. If now a fourth point D_1 be given corresponding to a fourth point D_2 and P_1 be any fifth point to which P_2 corresponds, the point in which D_2P_2 meets p_2 corresponds to the point in which D_1P_1 meets p_1 and is known. Therefore D_2P_2 is known, but the position of P_2 on it is indeterminate.

In like manner it may be shown that a homography is determined when four lines of one figure, no three of which are concurrent, are made to correspond to four lines of the other figure, no three of which are concurrent.

138. Vanishing lines. The equations of the vanishing lines of the homography are easily written down from equations (3) and (5) of Art. 136. For if x', y' are to be infinite we must have

$$A_3x + B_3y + C_3 = 0.$$

This then is the vanishing line of the figure ϕ. If x, y are to be infinite, then

$$(A_2B_3 - A_3B_2)\,x' + (A_3B_1 - A_1B_3)\,y' + A_1B_2 - A_2B_1 = 0,$$

and this gives the vanishing line of the figure ϕ'.

139. Reciprocal transformation or correlation. If in the equations (3), (4) and (5) of Art. 136 we interchange l', m' and x', y' we find, writing for shortness

$$\alpha_1 = B_2C_3 - B_3C_2,\ \ \alpha_2 = B_3C_1 - B_1C_3,\ \ \alpha_3 = B_1C_2 - B_2C_1,$$

with corresponding meanings for β's and γ's:

$$l' = \frac{A_1x + B_1y + C_1}{A_3x + B_3y + C_3}, \qquad m' = \frac{A_2x + B_2y + C_2}{A_3x + B_3y + C_3},$$

$$l = \frac{A_1x' + A_2y' + A_3}{C_1x' + C_2y' + C_3}, \qquad m = \frac{B_1x' + B_2y' + B_3}{C_1x' + C_2y' + C_3},$$

$$x = \frac{\alpha_1 l' + \alpha_2 m' + \alpha_3}{\gamma_1 l' + \gamma_2 m' + \gamma_3}, \qquad y = \frac{\beta_1 l' + \beta_2 m' + \beta_3}{\gamma_1 l' + \gamma_2 m' + \gamma_3},$$

$$x' = \frac{\alpha_1 l' + \beta_1 m' + \gamma_1}{\alpha_3 l' + \beta_3 m' + \gamma_3}, \qquad y' = \frac{\alpha_2 l' + \beta_2 m' + \gamma_2}{\alpha_3 l' + \beta_3 m' + \gamma_3}.$$

These equations may be shown as in Art. 136 to be the necessary equations of transformation in any one-one algebraic correspondence of plane figures in which lines correspond to points and points to lines. Clearly any pencil is homographic, and therefore equi-anharmonic, with the corresponding range. This transformation is therefore of the type discussed in Art. 63.

It is, however, much more general than this transformation; for the transformation by reciprocal polars is limited to figures in the same plane, whereas the present transformation is for any plane figures. Also in the transformation by reciprocal polars the same line p corresponds to the same point P whether P be considered as belonging to one figure or to the other. Whereas here, if the figures be taken coplanar and the axes of coordinates identical, if we put $x' = x$, $y' = y$ we do not in general obtain $l' = l$ or $m' = m$.

The present transformation is the most general case of a plane reciprocal transformation.

An obvious modification of the reasoning of Art. 137 will show that a correlation is determined when four points A_1, B_1, C_1, D_1 of one figure, no three of which are collinear, are made to correspond to four lines a_1, b_1, c_1, d_1 of the other figure, no three of which are concurrent.

For if P_1 correspond to p_2 we have

$$A_1\{B_1C_1D_1P_1\} = a_2\{b_2c_2d_2p_2\},$$

$$B_1\{A_1C_1D_1P_1\} = b_2\{a_2c_2d_2p_2\},$$

which determine a_2p_2 and b_2p_2, and therefore p_2.

And it is easy to show that if P_1 describes a straight line, p_2 passes through a point.

Two such figures may be said to be reciprocally homographic, or, more simply, reciprocal or correlative. The relation between them may be spoken of as a reciprocal homography or a correlation.

EXAMPLES VIII.

1. Prove that if an imaginary line l do not intersect its conjugate imaginary l', the line drawn from a real point P to meet l and l' is always real.

2. Show that the reciprocal elements of two conjugate imaginary elements are themselves conjugate imaginary when the reciprocal elements of real elements are real.

3. Show that conjugate imaginary elements project into conjugate imaginary elements when the projection is real.

4. Show that the square of the distance between two conjugate imaginary points is essentially negative.

5. A variable circle cuts a fixed circle at a constant angle a and passes through a fixed point O. If the points of intersection of this circle with the fixed circle be P, P', show that the ranges $[P]^2$, $[P']^2$ are homographic.

6. The coordinates of two points on a straight line are connected by the relation

$$\frac{1}{x} + \frac{1}{x'} = \frac{1}{f}.$$

Show that the points describe projective ranges.

7. The angles θ, θ' which two lines through a fixed origin make with an initial line are connected by the equation

$$\theta' = \frac{A\theta + B}{C\theta + D}.$$

Explain carefully why the two lines do not describe homographic pencils.

8. If the angles θ, θ' in the last question be connected by the relation

$$\sin\theta' = \frac{A\sin\theta + B}{C\sin\theta + D},$$

show that the lines do not describe homographic pencils.

9. A conic through four fixed points, two of which lie on a fixed conic s, meets s at P, P'. O is a fixed point on s. Prove that OP, OP' describe homographic pencils.

10. Through the vertex of a flat pencil planes are drawn perpendicular to the rays of the pencil. Show that the axial pencil so formed is homographic with the given flat pencil.

11. A conic through four points A, B, C, D meets fixed lines through A and B at P and Q. Show that P, Q describe homographic ranges.

12. If in a homographic relation the points (0, 0), (0, 1), (1, 0), (1, 1) in the plane of (x, y) correspond respectively to the points (0, 0), (1, 3), (2, 2), (2, 4) in the plane of (x', y'), show that the point (0, 2) in the plane of (x, y) corresponds to the point $(\frac{4}{3}, 4)$ in the plane of (x', y').

CHAPTER IX.

TRANSFORMATION OF PLANE FIGURES.

140. Any four coplanar points can be projected into any four coplanar points. Let A_1, B_1, C_1, D_1 be four points, no three of which lie in a straight line, in a plane α_1; and A_2, B_2, C_2, D_2 be four points, no three of which lie in a straight line, in a plane α_2. Through A_1 draw a plane α_3 not coincident with α_1. Project A_2, B_2, C_2, D_2 on to α_3 from a point S on A_1A_2 other than A_2. Let the projected points be A_1, B_3, C_3, D_3.

Let $(A_1B_1, C_1D_1) = E_1$; $(A_1B_3, C_3D_3) = E_3$.

Because the straight lines $A_1B_1E_1$, $A_1B_3E_3$ intersect, B_1, B_3, E_1, E_3 are coplanar. Therefore B_1B_3, E_1E_3 meet at a point U.

Through the line $A_1B_1E_1$ draw a plane α_4 not coincident with α_1. Let the projections of A_1, B_3, C_3, D_3, E_3 from U on to α_4 be A_1, B_1, C_4, D_4, E_1.

The points C_4, D_4, E_1 are collinear, since C_3, D_3, E_3 are collinear. Hence the lines $C_1D_1E_1$, $C_4D_4E_1$ are coplanar. $\therefore$ C_1C_4, D_1D_4 meet at some point V.

Projecting $A_1B_1C_4D_4$ from V on to α_1 we obtain $A_1B_1C_1D_1$. Thus we may pass from A_2, B_2, C_2, D_2 to A_1, B_1, C_1, D_1 by three projections.

Similarly any four coplanar lines a_1, b_1, c_1, d_1, no three of which pass through a point, can always be projected into any four coplanar lines a_2, b_2, c_2, d_2, no three of which pass through a point. For in this case the four points a_1b_1, b_1c_1, c_1d_1, d_1a_1 are distinct and no three of them are collinear, and the same holds of the four points a_2b_2, b_2c_2, c_2d_2, d_2a_2. These two sets of four points are therefore projective by the first part of the present article, and the lines which join them are likewise projective which proves the proposition.

141. Every plane homography is a projective transformation and conversely. For consider any plane homography. Take two corresponding tetrads such that no three points of each are collinear, and construct a projective transformation in which these are corresponding tetrads. Since both homography and projection preserve cross-ratio constant, the construction given in Art. 137 for finding the point P_2 corresponding to any given point P_1 applies to both the projective and homographic transformations. These two transformations therefore determine the same correspondence between the two figures, that is, the given homography is identical with the projective transformation.

Conversely every projective transformation is homographical, for it is a one-one algebraic transformation in which points correspond to points and straight lines to straight lines.

It follows from Art. 137 that two corresponding tetrads of points or lines entirely determine the projective correspondence between two planes.

142. Deductions from the above. If we are given three points A_1, B_1, C_1 on a conic s_1 and three points A_2, B_2, C_2 on a conic s_2 the conic s_1 can always be projected into the conic s_2 and at the same time the three points A_1, B_1, C_1 into the three points A_2, B_2, C_2.

Draw the tangents to s_1 at A_1, B_1 meeting at D_1 and the tangents to s_2 at A_2, B_2 meeting at D_2. Project the four points A_1, B_1, C_1, D_1 into the four points A_2, B_2, C_2, D_2. Then s_1 projects into a conic which touches D_2A_2 at A_2, D_2B_2 at B_2 and passes through C_2. But this conic must be s_2 for two pairs of coincident points and another point determine a conic uniquely.

In like manner if a_1, b_1, c_1 be three tangents to a conic s_1; a_2, b_2, c_2 three tangents to a conic s_2, let d_1 be the chord of contact of a_1b_1, d_2 the chord of contact of a_2b_2. Project $a_1b_1c_1d_1$ into $a_2b_2c_2d_2$. Then s_1 projects into a conic touching a_2 at a_2d_2, b_2 at b_2d_2 and touching also d_2. And this conic can be none other than s_2.

These two results show that two ranges or pencils of the second order can always be actually projected into one another so that any two given triads correspond. Equi-anharmonic ranges of the second order are therefore actually projective, which justifies the name given to them in Chapter VII.

Notice that the condition that two given ranges of the second order, or two given pencils of the second order, shall correspond, determines entirely the projective relation between the two planes.

143. Self-corresponding elements of two coplanar projective figures. Clearly two coplanar projective figures cannot have more than three non-collinear self-corresponding points or more than three non-concurrent self-corresponding lines; or else it would follow from Art. 141 that they coincided altogether.

Consider any point O of the plane. To O considered as a point of the first figure let O_2 correspond; to O considered as a point of the second figure let O_1 correspond. The corresponding rays of the first and second figures through O, O_2 respectively sweep out two projective pencils: hence their intersection describes a conic u. The corresponding rays of the first and second figures through O_1, O respectively sweep out two projective pencils: hence their intersection describes a conic v. u, v have an intersection O: in general they will have three other intersections P, Q, R.

Now OP of the first figure corresponds to O_2P of the second and O_1P of the first figure corresponds to OP of the second. Hence (OP, O_1P), i.e. P, of the first figure, corresponds to (O_2P, OP), i.e. P, of the second figure. Thus P, and therefore also Q, R, are the three self-corresponding points. One of these is always real, since of the four intersections O, P, Q, R of u, v, O is always real.

The three self-corresponding lines are clearly PQ, QR, RP. They may also be obtained by considering a line x and its two correspondents x_1, x_2. Joining corresponding points on x_1, x and on x, x_2 we obtain two conics touching x. Their three other common tangents p, q, r are self-corresponding lines. Since x is a real common tangent, a second common tangent must also be real, so that one self-corresponding line is real. Clearly if P is the only real self-corresponding point, QR is the only real self-corresponding line, although Q, R themselves are not real.

In general on a self-corresponding line p there are only two self-corresponding points, Q, R being the self-corresponding points of the projective ranges formed by corresponding points on p. If however a third self-corresponding point on p exists, then every point of p is self-corresponding. We have then the case of figures in plane perspective, p is the axis of collineation and the self-corresponding point P not on p is the pole of perspective.

The conics u and v then coincide, each of them breaking up into the lines p and OP. All the points of p are self-corresponding: but the only self-corresponding points of OP are P and the point where OP meets p.

144. Two reciprocal transformations are equivalent to a projective transformation. Consider two plane reciprocal figures ϕ_1, ϕ_2.

Now take a figure ϕ_3 reciprocal with ϕ_2. ϕ_1 and ϕ_3 now correspond point by point and line by line and since the correspondence between elements of ϕ_1 and ϕ_2 is one-one and algebraic, and that between elements of ϕ_2 and ϕ_3 is one-one and algebraic, the correspondence between elements of ϕ_1 and ϕ_3 is also one-one and algebraic.

Therefore the figures ϕ_1 and ϕ_3 are homographic and therefore projective.

145. Any reciprocal transformation is equivalent to a projective transformation and a transformation by reciprocal polars. For let ϕ_1 and ϕ_2 be given reciprocal figures. Let ϕ_3 be the reciprocal polar figure of ϕ_2 with regard to any conic. Then by the last Article ϕ_1 and ϕ_3 are projective. Thus a projective transformation transforms ϕ_1 to ϕ_3 and the transformation by reciprocal polars transforms ϕ_3 to ϕ_2.

146. Locus of incident points and envelope of incident lines of two coplanar reciprocal figures. If two reciprocal figures ϕ, ϕ' be coplanar, we proceed to find the condition that a point and its corresponding line shall be incident.

If P be a point on its corresponding line p', P being considered as belonging to figure ϕ, it also lies on its corresponding line when considered as belonging to figure ϕ'. For let $P = Q'$. Then, since P, i.e. Q', lies on p', q passes through P, i.e. through Q'.

In like manner if a line passes through its corresponding point in one figure, it passes also through its corresponding point in the other figure.

Such points and lines may be called incident points and lines.

Let u be any line of one figure, U' its corresponding point of the other. Let P be any point of u, its corresponding line p' passes through U' and meets u at P'. The range $[P]$ is homographic with the pencil $[p']$ and therefore projective with the range $[P']$. The two ranges $[P]$, $[P']$ have therefore two self-corresponding points S, T which are such that they lie on their corresponding lines.

The locus of such points has therefore two points of intersection with any straight line. Hence it is a conic s_1 by Art. 134.

In like manner if p be any line through a point U, P' its corresponding point on u', and $p' = UP'$ the pencils $[p]$, $[p']$ are projective and their two self-corresponding rays are the tangents from U to the envelope of incident lines. This envelope is then a curve of the second class and therefore a conic s_2.

The conics s_1 and s_2 are clearly, from their definition as locus of incident points and envelope of incident lines, corresponding curves in the two figures. Hence to a point of one corresponds a tangent of the other and conversely.

Any tangent p to s_2 meets s_1 at two points P and P' which are obviously the two points corresponding to p in the two figures. To the point of contact of p with s_2, considered as a point of the first figure, corresponds a tangent to s_1 through P', i.e. the tangent at P', and in like manner to this same point of contact of p with s_2, considered as a point of the second figure, corresponds the tangent to s_1 at P.

In like manner the two corresponding lines to a point P on s_1 are the two tangents from P to s_2, and the two corresponding points of the tangent at P to s_1 are the points of contact of the tangents from P to s_2.

Consider now what happens at a point of intersection of s_1 and s_2. If P be such a point the two tangents from P to s_2 are coincident with the tangent to s_2 at P; call this tangent p. Then P corresponds to p considered as a line of either figure, and P lies on p. Hence this tangent p touches s_1 at P (since its two intersections with s_1 are its correspondents in either figure and these coincide at P).

The two conics s_1 and s_2 therefore touch at each of their intersections. They have then double contact. If P and Q be the two points of contact, p, q the common tangents at P, Q; P, Q are correspondents of p, q in either figure. Hence PQ corresponds to pq in either figure. Thus the pole of the chord of contact corresponds to the chord of contact in either figure.

If the transformation be one by reciprocal polars the two conics s_1, s_2 coincide with the base conic, for every point of the base conic lies on its polar and every tangent to the base conic passes through its pole.

EXAMPLES IX.

1. Given the three self-corresponding of two projective coplanar figures and a pair of corresponding points, give a construction for the point corresponding to any given point, and also for the vanishing lines.

2. Given three pairs of corresponding points of two homographic plane figures and one of the self-corresponding lines, construct the intersection of the other two self-corresponding lines.

3. Show how to set up a one-one correspondence of a plane into itself such that a conic in the plane is transformed into itself and three assigned points of it into three other assigned points of it.

4. Show that the reciprocal polar figure of a circle s with regard to another circle c is a conic, one of whose foci is the centre of c, and find a construction for the other focus. Show also that the conic is an ellipse, parabola, or hyperbola according as the centre of c lies inside, on, or outside s.

[The student should note that this furnishes a method of discovering focal properties of the conic from properties of the circle. It will be an instructive exercise for him to deduce the results of Arts. 98, 102, 104 in this way.]

5. The polar reciprocal of a circle, taken with regard to a rectangular hyperbola, is a conic of which the centre of the rectangular hyperbola is a focus.

6. Prove that a conic is the polar reciprocal of its auxiliary circle with regard to a circle of imaginary radius whose centre is a focus.

7. If one conic s is its own polar reciprocal for another conic t then the conic t is its own polar reciprocal for the conic s.

[Show that the conics have double contact at A, B and that if C be the common pole of AB then if a ray through C meet t at U, T the tangent at U is the polar of T with regard to s.]

8. Show that given two points A, B of a conic s, a conic t can be found having double contact with s at A, B such that s is its own reciprocal with regard to t.

CHAPTER X.

INVOLUTION.

147. If in two cobasal homographic like forms one pair of elements correspond doubly, all pairs correspond doubly. Let P, P' be two corresponding elements (denoted by Roman capitals, but not restricted to mean points) of two homographic like forms ϕ, ϕ' having the same base.

Then in general if P be considered as an element of ϕ' the element of ϕ which then corresponds to P is not P', but some other point.

It may, however, happen that P' corresponds to P, whether P be considered as belonging to ϕ or as belonging to ϕ'. P and P' are then said to correspond doubly.

In this case every other pair of corresponding elements Q, Q' also correspond doubly. For since by Art. 21 a cross-ratio is not altered if we interchange two of its elements, provided the other two be also interchanged,

$$\{PP'QQ'\} = \{P'PQ'Q\}.$$

But by hypothesis $PP'Q$, $P'PQ'$ are corresponding triads of ϕ, ϕ' respectively. Hence the above equation expresses the fact that to Q' of ϕ corresponds Q of ϕ' or Q, Q' correspond doubly.

148. Involution. Two cobasal homographic like forms, in which every element corresponds doubly, are said to be in *involution*, or to form an involution on their base. The corresponding elements are spoken of as mates in the involution.

149. Two pairs of mates determine an involution. Let (P, P'), (Q, Q') be the two pairs of mates. Then the triads $PP'Q$, $P'PQ'$ define two homographic forms which are in involution since one pair of elements, namely P, P', correspond doubly. The involution is therefore determined.

Note that *one* pair of mates is insufficient; for *two* pairs of

corresponding points (P, P'), (P', P) are not enough to determine two homographic forms.

150. Double elements. Since two homographic cobasal like forms have two self-corresponding elements, an involution will have two self-corresponding mates, which may be real or imaginary.

These are called the double elements of the involution. Since a double element is equivalent to a pair of mates, an involution is entirely given by its double elements.

An involution whose double elements are real is said to be *hyperbolic*; one whose double elements are imaginary is said to be *elliptic*.

151. Any pair of mates are harmonically conjugate with regard to the double elements. For let (P, P') be a pair of mates; A, B the double elements. Then the elements $APBP'$ correspond to $AP'BP$ or

$$\{APBP'\} = \{AP'BP\}.$$

The set $APBP'$ are therefore equi-anharmonic with themselves, P and P' being interchanged: therefore (Art. 27) P and P' are harmonically conjugate with regard to A, B.

152. Involution on a straight line. Centre of involution. Consider now the case of an involution on a straight line. Let O be the mate of the point O'^{∞} at infinity on the straight line. O is called the centre of involution. If (P, P'), (Q, Q') be two pairs of mates, we have

$$\{OPO'^{\infty}Q\} = \{O'^{\infty}P'OQ'\},$$

or
$$\frac{OP \cdot O'^{\infty}Q}{OQ \cdot O'^{\infty}P} = \frac{O'^{\infty}P' \cdot OQ'}{O'^{\infty}Q' \cdot OP'}, \qquad \text{i.e. } \frac{OP}{OQ} = \frac{OQ'}{OP'},$$

therefore $OP \cdot OP' = OQ \cdot OQ' =$ constant for the involution.

If Q, Q' coincide with one of the double points A, B we have

$$OP \cdot OP' = OA^2 = OB^2.$$

In a hyperbolic involution A, B are real, thus OA^2, OB^2 are positive and $OP \cdot OP'$ is positive. Conversely, if $OP \cdot OP'$ is positive, A, B are real. Therefore in an elliptic involution $OP \cdot OP'$ is negative and conversely.

Since $OA^2 = OB^2$, O is midway between the double points.

153. Relation between the mutual distances of six points in involution. Let (A_1, A_2), (B_1, B_2), (C_1, C_2) be three pairs of mates of an involution.

Then $\{A_1A_2B_1C_1\} = \{A_2A_1B_2C_2\}$,

or, writing out the cross-ratios,

$$\frac{A_1A_2 \cdot B_1C_1}{A_1C_1 \cdot B_1A_2} = \frac{A_2A_1 \cdot B_2C_2}{A_2C_2 \cdot B_2A_1}.$$

Cancelling out $A_1A_2 \, (= -A_2A_1)$ and re-arranging, we have

$$B_1C_1 \cdot C_2A_2 \cdot A_1B_2 = -B_2C_2 \cdot C_1A_1 \cdot A_2B_1.$$

Now since mates in an involution have symmetrical properties, we may, in this result, interchange the suffixes 1 and 2 belonging to any letter A, B or C, and the result is still true. It may therefore be stated generally in the following form,

$$(BC \cdot CA \cdot AB)_{1,2} = -(BC \cdot CA \cdot AB)_{2,1},$$

where $(BC \cdot CA \cdot AB)_{1,2}$ indicates any distribution of suffixes such that a 1 and a 2 go to each letter.

154. Coaxial circles. The common chord CD of two circles is called their *radical axis*. If the circles do not cut in real points their intersections must be conjugate imaginary points and the radical axis is still a real line (Art. 125). Any point S on the radical axis possesses the property that the tangents to the two circles from S are equal. For the square of each of these tangents is $SC \cdot SD$. The property remains true even when the radical axis does not cut the circles in real points, from the general principles explained when imaginary points were introduced.

A set of circles passing through two fixed points C, D are called coaxial circles: any pair of them have the same line CD for their radical axis.

155. Coaxial circles determine an involution on any straight line. Let C, D be the common points of the circles, x the straight line. Let O (Fig. 43) be the point where CD meets x, P, P' the points where any circle of the system meets x, then $OP \cdot OP' = OC \cdot OD =$ constant for all the circles of the system.

Thus the circles determine on x an involution of which O is the centre. The double points of this involution are the points of contact of the circles through C, D and touching x.

156. Construction for the double points of an involution. The last result gives a method for constructing the double points of an involution on a straight line x, when two pairs of mates (P, P'), (Q, Q') are given. Describe any two circles (Fig. 43 *a*) passing through PP', QQ' respectively. These

circles can always be taken of so large a radius that they intersect in real points C, D. CD then meets x at the centre O of involution and a circle with centre O and radius equal to the tangent from O to either circle meets x at the two double points A, B. For $OA^2 = OC \,.\, OD = OP \,.\, OP' = OQ \,.\, OQ'$.

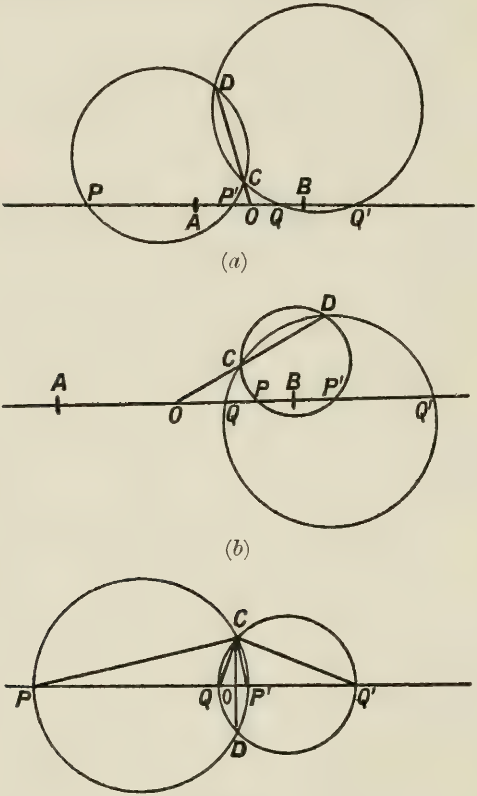


Fig. 43.

157. An involution is elliptic or hyperbolic according as a pair of mates are, or are not, separated by any other pair of mates. Consider first an involution on a straight line. Let (P, P'), (Q, Q') be any two pairs of mates.

Construct the double points of the involution by the method of the last Article.

Then if (Fig. 43 a) the segments PP', QQ' do not overlap, that is, if the mates (P, P') are not separated by the mates (Q, Q'), the two circles intersect at points C, D on the same side of x. O is outside CD and therefore outside both circles. The tangents from O to the circles, and therefore the double points, are real and the involution is hyperbolic.

If one of the segments PP' lies entirely inside the other QQ' the same result follows (see Fig. 43 b). In this case also one pair of mates are not separated by the other. For Q, Q' are not then looked upon as separated by P and P'; Q and Q' may be connected by a continuous sequence of points of the line, passing through the point at infinity but not including P or P'. For mates to be considered "separated" they must be in an order such as $PQP'Q'$.

But if the segments PP', QQ' overlap (Fig. 43 c), then (P, P') are separated by (Q, Q'). The circles through P, P' and Q, Q' intersect at points C, D on opposite sides of x. O lies inside both circles. No real tangents can be drawn from O to the circles, and the involution is elliptic.

If we now consider an involution flat pencil, or an involution axial pencil, these determine on any straight line which meets them an involution range (since two cobasal homographic pencils, flat or axial, determine homographic ranges on any transversal and elements corresponding doubly in the pencils give elements corresponding doubly in the ranges). Also if mates are separated in the pencil, they are so in the range which is a section of the pencil; and the double rays or planes of the one pass through the double points of the other. The involution pencil and the involution range are therefore elliptic and hyperbolic together. It follows that if in an involution pencil (flat or axial) a pair of mates are separated by any other pair of mates the involution is elliptic. If they are *not* so separated the involution is hyperbolic.

The student should have no difficulty in proving, by precisely similar reasoning, that the same theorem holds also for an involution of points on a conic and for an involution of tangents to a conic.

158. Involution flat pencil. In an involution pencil there is no special ray corresponding to the centre of an involution range, for no ray is the analogue of the point at infinity.

If OA, OB are the double rays, (OP, OP') a pair of mates,

then cutting the pencil by a straight line parallel to OP', which meets the double rays at A and B (Fig. 44), AB is bisected at P by OP, since OP, OP' are harmonic conjugates with regard to OA, OB and therefore P and the point at infinity on OP' are harmonic conjugates with regard to A, B. Hence if the parallelogram whose sides are OA, OB be completed, its diagonals are parallel to a pair of mates.

If the double rays are at right angles, every such parallelogram is a rectangle. Its diagonals are equally inclined to the sides of the rectangle, therefore if the double rays are at right angles, they bisect the angles between any pair of mates.

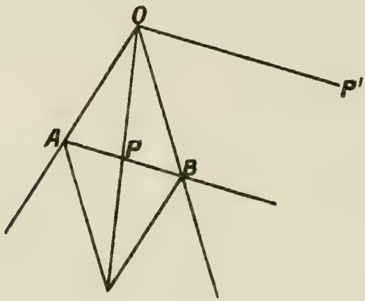


Fig. 44.

159. Relation between six rays of an involution. Proceeding as in Art. 153, we have, if $(OA_1,\ OA_2)$, $(OB_1,\ OB_2)$, $(OC_1,\ OC_2)$ are three pairs of mates of an involution pencil,

$$O\{A_1A_2B_1C_1\} = O\{A_2A_1B_2C_2\},$$

and, using the expression for the cross-ratio of a pencil in terms of the angles made by the rays (Art. 22),

$$\frac{\sin A_1OA_2 \,.\, \sin B_1OC_1}{\sin A_1OC_1 \,.\, \sin B_1OA_2} = \frac{\sin A_2OA_1 \,.\, \sin B_2OC_2}{\sin A_2OC_2 \,.\, \sin B_2OA_1},$$

whence

$$\sin B_1OC_1 \,.\, \sin C_2OA_2 \,.\, \sin A_1OB_2$$
$$= -\sin B_2OC_2 \,.\, \sin C_1OA_1 \,.\, \sin A_2OB_1,$$

and interchanging suffixes as in Art. 153 we have the general result

$$(\sin BOC \,.\, \sin COA \,.\, \sin AOB)_{1,2}$$
$$= -(\sin BOC \,.\, \sin COA \,.\, \sin AOB)_{2,1},$$

where the suffixes 1, 2 on the left-hand side indicate that a 1 and a 2 are to be assigned to each of the three letters A, B, C, the order being arbitrary.

160. Rectangular involution. Rays at right angles through a point O determine an involution pencil through O. For clearly the relation between a ray and its perpendicular is one-one and algebraic. Thus the rays generate two homographic pencils through O. Also if p be perpendicular to p', so is p' perpendicular to p. Therefore all pairs of rays correspond doubly. They accordingly form an involution. Such an involution is clearly elliptic, for its double rays must be at right angles to themselves, a condition which cannot be satisfied by any real lines.

Since an involution is determined by two pairs of mates, if two pairs of mates of an involution pencil are rectangular, the involution is a rectangular involution. For a rectangular involution clearly satisfies the required conditions and there can be only one involution which does so.

161. In any plane through a straight line there are two points, symmetrically situated with regard to the line, from which an involution range on it can be projected as a rectangular involution. For take the circles of Art. 157 to be the circles on PP', QQ' as diameters (Fig. 43 c). These intersect at two points C, D symmetrically situated with regard to the base x of the involution. The pencil obtained by joining either C or D to the points of the involution range on x must be an involution pencil, mates in which pass through mates in the involution range. But such an involution pencil has two pairs of mates at right angles, for CP, CP' are perpendicular and CQ, CQ' are perpendicular, since the angle in a semicircle is a right angle. Hence it is a rectangular involution. Similarly the involution pencil through D is a rectangular involution.

These points C and D are real only if the mates P, P' are separated by the mates Q, Q', that is, if the original involution range is elliptic. In the other cases, where the segment PP' is either entirely outside, or entirely inside, QQ', the circles on PP' and QQ' as diameters do not cut in real points.

162. Common mates of two involutions on the same base. First of all consider two involution ranges on the same straight line. We require to find two points of the line which are mates in both involutions.

Let A_1, B_1 be the double points of one involution, A_2, B_2 the double points of the other. Then if P, P' are mates in both involutions, (P, P') are harmonically conjugate with regard to both (A_1, B_1) and (A_2, B_2); that is, they are the double points of the involution determined by the two pairs (A_1, B_1), (A_2, B_2). They can therefore be found geometrically by the construction of Art. 156, provided the two given involutions are hyperbolic. In this case the pair of common mates are imaginary or real, according as the double points of one involution are, or are not, separated by the double points of the other.

If, however, one involution is hyperbolic and the other elliptic,

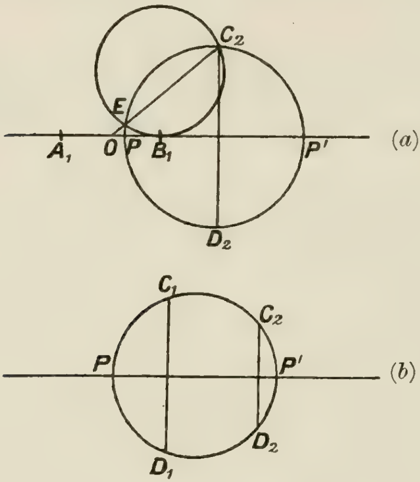


Fig. 45.

let (A_1, B_1) (Fig. 45 a) be the double points of the hyperbolic involution and C_2, D_2 the two points from which the elliptic involution is rectangularly projected. Then the points where the base x is cut by any circle through C_2, D_2 are mates of the elliptic involution, for by symmetry such a circle has x for a diameter and its points of intersection with x therefore subtend a right angle at C_2.

Construct the circle which touches x at B_1 (or A_1) and passes through C_2. Let O be the middle point of A_1B_1. O is the centre of the hyperbolic involution. Join OC_2, cutting this

circle again at E. Through C_2, E, D_2 draw a circle meeting x at P, P'. Then P, P' are the points required.

For P, P' being on a circle through C_2, D_2 are mates in the elliptic involution.

Also $OP \cdot OP' = OE \cdot OC_2 = OB_1^2$, $\therefore$ P, P' are mates in the hyperbolic involution.

Finally if both involutions are elliptic let C_1, D_1 and C_2, D_2 be the points from which they are rectangularly projected (Fig. 45 *b*). Then by symmetry a circle will pass through the four points C_1, D_1, C_2, D_2. The points P, P' where this circle meets x are mates in both involutions.

Note that in the last two constructions the points found are always *real.*

If now it be required to find the mates common to two concentric involution pencils of vertex O, we cut the pencils by a transversal. We obtain two involution ranges and find their common mates as above. The rays joining these common mates to O are the common mates of the given involution pencils.

In like manner the problem of finding the common mates of two cobasal involutions of any kind is always reducible to the same problem for involution ranges on a straight line.

The problem has a real solution in all cases except when the two involutions are both hyperbolic and the double elements of one are separated by the double elements of the other.

In particular every involution flat pencil has always one pair of real mates at right angles and one only, these being the common mates of the given involution with the rectangular involution through the same vertex. These must be real, since the rectangular involution is elliptic.

163. Involutions of conjugate elements with regard to a conic. The two collinear projective ranges formed by associating with each point of a line its conjugate point with regard to a conic (Art. 55) define an involution, since, from the symmetry of the conjugate relation, two corresponding points correspond to each other doubly. The double points of this involution are the points where the straight line meets the conic.

Similarly conjugate lines through a point form an involution of which the double rays are the tangents from the point.

In particular conjugate diameters form an involution, of which the double rays are the asymptotes.

Since the involution of conjugate diameters has one real pair of mates at right angles and one only, we obtain a new proof of the theorem of Art. 68 that a conic has one, and only one, pair of axes.

164. The circular points at infinity. Consider the two (imaginary) points in which the line at infinity i^∞ in a plane meets any circle in the plane. Since the pole of i^∞ is the centre C of the circle the involution of conjugate points on i^∞ is given by the intersection of i^∞ with the involution of conjugate rays through C. But since conjugate diameters of a circle are at right angles (Art. 57) the latter involution is the rectangular involution through C.

The two intersections Ω, Ω' of i^∞ with the circle are therefore the double points of the involution in which the rectangular involution through C meets i^∞.

But if we take any other point O in the plane and join O to the points of the involution on i^∞ we obtain an involution through O whose rays are parallel to the corresponding rays of the involution through C. The involution through O is therefore also rectangular. Thus the double rays of all rectangular involutions pass through the same two points Ω, Ω' at infinity. These points Ω, Ω' are therefore determined quite independently of the particular circle chosen. Hence all circles pass through the same two points Ω, Ω'.

Conversely every conic which passes through Ω, Ω' is a circle. For let s be such a conic and let A, B, C be any three other points on s. Describe the circle c through A, B, C. Then it passes through Ω, Ω'. c and s have five points A, B, C, Ω, Ω' common and therefore coincide.

For these reasons the points Ω, Ω' are called the *circular points at infinity*. Being the intersections of a real line (the line at infinity) with a real curve, they are conjugate imaginary points by Art. 125.

Two interesting cases of circles arise when the conic through Ω, Ω' degenerates into a line-pair. If Ω, Ω' are on the same component of the pair, the latter consists of the line at infinity and a straight line at a finite distance. Thus any straight line, together with the line at infinity, may be regarded as forming a circle of infinite radius.

If Ω, Ω' be on different components of the pair we see that any pair of lines through Ω, Ω' form a circle.

If their point of intersection P be real, every line through P is a tangent to the curve at P (see Art. 44) and the circle is called a *point-circle*.

165. The circular lines. The lines joining any point of the plane to Ω, Ω' are called the *circular lines* through the point. If the point be real, the circular lines through it are conjugate

imaginaries. From the last Article the circular lines through a point are the double rays of the rectangular involution through the point.

Hence any pair of lines at right angles are harmonically conjugate with regard to the circular lines through their intersection.

It follows that if in any involution pencil the circular rays are mates, the double rays are at right angles. Conversely if the double rays are at right angles the circular rays are mates.

166. The arms of an angle of given magnitude determine with the circular lines through its vertex a constant cross-ratio. Consider an angle of given magnitude rotating about its vertex O. Its arms trace out two directly equal concentric flat pencils of which the self-corresponding rays are by Art. 116 parallel to the asymptotes of a circle, that is, they are the circular lines through O. Thus if POP', QOQ' be any two positions of the angle, (OP, OP'), (OQ, OQ') are two pairs of corresponding rays; they determine therefore the same cross-ratio with the circular lines through O (Art. 113).

If on the other hand two angles with different vertices O, O' have their arms parallel, the parallel arms and the circular lines through O, O' determine the same range on the line at infinity. They form two perspective flat pencils and the cross-ratios are the same.

Combining the above two results, if an angle of given magnitude be moved about in its own plane anyhow, it defines a fixed cross-ratio with the circular lines through its vertex.

The converse theorem that, if a moving angle determine with the circular lines through its vertex a constant cross-ratio, the magnitude of the angle is fixed, is readily proved.

167. The circular points are conjugate with regard to any rectangular hyperbola. For in a rectangular hyperbola the double rays of the involution of conjugate diameters are at right angles. Therefore the circular lines through the centre are conjugate. The points where they meet the polar of the centre (i.e. the line at infinity) are therefore also conjugate with regard to the hyperbola. But these are the circular points Ω, Ω'.

Conversely if Ω, Ω' are conjugate points with regard to the hyperbola, the circular lines through the centre are conjugate lines and the asymptotes are at right angles.

168. The orthoptic circle. Consider the pencils of conjugate rays with respect to any conic s through the circular points Ω, Ω'. These pencils are projective by Art. 55. Their product is therefore a conic passing through Ω, Ω', that is, a circle.

Let P be any point on this circle. Then $P\Omega$, $P\Omega'$ being lines through P conjugate with regard to s are harmonically conjugate with regard to the two tangents from P to s (Art. 53). Therefore these two tangents are at right angles (Art. 165).

Conversely if these two tangents are at right angles $P\Omega$, $P\Omega'$ are mates in the involution of conjugate rays through P, and P lies on the product of the conjugate pencils through Ω, Ω'. We have then the theorem:

The locus of intersections of tangents to a conic at right angles is a circle.

The circle is called the *orthoptic circle* of the conic, from the property that at any point of it the conic subtends a right angle. It is also called the *director circle*, by analogy with its degenerate case when the conic is a parabola, when the locus of intersections of tangents at right angles is the directrix (Art. 106). The explanation of this from our point of view is that in the case of the parabola $\Omega\Omega'$ touches the curve and is therefore a self-corresponding ray of the conjugate pencils through Ω, Ω'. These are accordingly perspective and the locus breaks up into $\Omega\Omega'$ (the line at infinity) and another straight line, which is the directrix.

The orthoptic circle is concentric with the conic. For the tangent at Ω to the orthoptic circle is the line through Ω conjugate to $\Omega\Omega'$ with regard to the conic (Art. 39). It must therefore pass through the pole of $\Omega\Omega'$, i.e. through the centre of the conic. Similarly the tangent at Ω' to the orthoptic circle passes through the centre of the conic. The pole of $\Omega\Omega'$ with regard to the circle (i.e. the centre of the circle, $\Omega\Omega'$ being the line at infinity) is thus the centre of the conic.

The radius of the orthoptic circle is immediately found by drawing the (perpendicular) tangents at the extremities of the axes. The semi-diagonal of the rectangle so formed is the radius required. It is $\sqrt{CA^2 + CB^2}$. In the hyperbola $CB^2 = -CB_1^2$, so the radius of the orthoptic circle $= \sqrt{CA^2 - CB_1^2}$. If $CB_1^2 > CA^2$ the orthoptic circle is imaginary. If $CB_1^2 = CA^2$, or the hyperbola is rectangular, it shrinks into a point at the centre. Thus the only real perpendicular tangents to a rectangular hyperbola are the asymptotes.

169. The four foci of a conic. By the definition of a focus the involution of conjugate lines through it is rectangular. Thus the tangents from a focus to the conic, being the double rays of such an involution, are the circular lines through the focus and pass through Ω, Ω'. Conversely a point F which is the intersection of tangents from Ω, Ω' must be a focus, for the double rays of the involution of conjugate rays through F will be the tangents from F, namely $F\Omega$, $F\Omega'$. But these being the circular lines, the involution defined by them must be rectangular, or F is a focus.

Since two tangents t_1, t_2 can be drawn to a conic from Ω and two tangents t_1', t_2' can be drawn from Ω', a conic will have *four* foci, namely t_1t_1', t_1t_2', t_2t_1', t_2t_2'. Of these two are real and two imaginary, as follows. Take one tangent t_1 from Ω. This being an imaginary line in a real plane, has a real point F_1 on it (Art. 126). The other tangent from F_1 must be a conjugate imaginary line to t_1, for two imaginary tangents from a real point to a real conic must be conjugate imaginaries, as can be shown from reasoning similar to that used in Art. 125 to prove that intersections of a real line and a real conic are conjugate imaginaries.

This other tangent from F_1, being a conjugate imaginary to t_1, i.e. to $F_1\Omega$, must be $F_1\Omega'$. Call it then t_1 Let t_2 be the other tangent from Ω. If F_2 be the real point on it, then $F_2\Omega' = t_2'$, and t_2, t_2' are conjugate imaginary lines. F_1, F_2 are the two real foci of the curve. t_1t_2', t_2t_1', which we may call F_3 and F_4, are the intersections of non-conjugate imaginary lines and are imaginary points. They are, however, themselves conjugate imaginary points being intersections of two conjugate imaginary pairs (Art. 125). Hence F_3F_4 is a real line.

But by Art. 61 the diagonal triangle of the complete quadrilateral $t_1t_1't_2t_2'$ circumscribed to the conic is self-polar with regard to the conic. But the sides of this diagonal triangle are F_1F_2, F_3F_4, $\Omega\Omega'$. The meet of F_1F_2, F_3F_4 is therefore the pole of $\Omega\Omega'$, i.e. the centre C of the conic; F_3F_4, F_1F_2 are then conjugate diameters. By the harmonic property of the complete quadrangle $F_1F_3F_2F_4$ the two sides of the diagonal triangle through C, viz. $C\Omega$, $C\Omega'$, are harmonically conjugate to the two sides of the quadrangle through C, namely F_1F_2, F_3F_4. $C\Omega$, $C\Omega'$ being circular lines F_1F_2, F_3F_4 are perpendicular and so must be axes. The two imaginary foci therefore lie on what we have called hitherto the non-focal axis of the curve.

170. Confocal conics. If two foci F_1, F_2 of a conic be

given, the other foci F_3, F_4 are determined. For they are the remaining vertices of the complete quadrilateral formed by the four lines $F_1\Omega$, $F_1\Omega'$, $F_2\Omega$, $F_2\Omega'$.

In particular, conics which have the same two real foci have all their foci the same. Such conics are called confocal conics. They touch four fixed lines, namely the sides of the quadrilateral mentioned above.

171. The circular points are foci of a parabola. In the case of a parabola the line at infinity $\Omega\Omega'$ is a tangent. Thus t_2, t_2' coincide with $\Omega\Omega'$. The quadrilateral of tangents from Ω, Ω' reduces therefore to a triangle. F_1, i.e. $(t_1 t_1')$, remains as the only real focus of the curve at a finite distance, F_2 is the point of contact of the line at infinity, i.e. the point at infinity on the axis. F_3 and F_4 become intersections of t_1 and t_1' with the line at infinity, that is, they coincide with Ω, Ω' which are thus foci of the curve.

We have therefore an exception to the theorem of the last Article, for the giving of Ω, Ω' does not here determine the other foci.

172. Imaginary projections. By means of the circular points a number of important theoretical results in projection can be deduced.

Thus any two conics can always be projected simultaneously into circles.

For let A, B be any two of the intersections of such conics. Then by projecting A, B into the circular points in any plane, the conics are projected into circles.

This result is of great importance, since it enables us to apply to a pair of conics any projective theorem proved for a pair of circles.

This projection of two given points into the circular points is of course imaginary if the two given points are real. If the two given points are conjugate imaginary points, they will in general be given as the intersections of a straight line x with a conic s, when x and s do not cut in real points. Take O the pole of x with regard to s and two pairs (OP, OP'), (OQ, OQ') of conjugate lines through O with regard to s. Project x to infinity and the angles POP', QOQ' into right angles (Art. 19). O projects into the centre of the conic and (OP, OP'), (OQ, OQ') into pairs of conjugate diameters at right angles, i.e. into axes. But since a conic with more than one pair of axes must be a circle, s projects into a circle and its intersections with x into the inter-

sections of a circle with the line at infinity, that is, into the circular points. Thus a real projection transforms a pair of conjugate imaginary points into the circular points.

Again, two conics can always be projected simultaneously into rectangular hyperbolas. For take AB a common chord of the conics. On AB take any two points C, D harmonically conjugate with regard to A, B. Then C, D are conjugate points with regard to both conics. Project C, D into the circular points in any plane, the conics, by Art. 167, project into rectangular hyperbolas.

Also two conics can always be projected into two confocal conics, by taking two opposite vertices of the complete quadri-

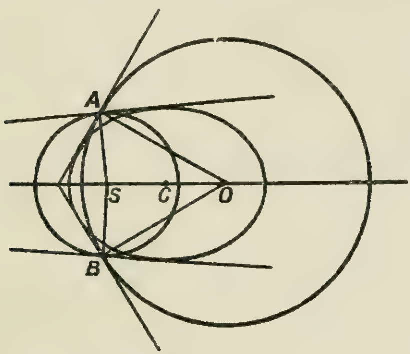


Fig. 46.

lateral formed by their common tangents and projecting these vertices into the circular points. The two projected conics have the same tangents from the circular points and are therefore confocal. Accordingly all projective properties of confocal conics are properties of any pair of conics.

173. The eight tangents to two conics at their four common points touch a conic. We will take an example of the deduction of theorems for two conics from theorems for two circles.

Let two circles whose centres are S, O (Fig. 46) intersect at A and B. A and B are symmetrically situated with regard to SO. Let C be the middle point of SO. The circle whose centre is C and which passes through A also passes through B. Con-

struct the conic having S, O for foci and the circle with centre C and radius CA for auxiliary circle. This conic touches the tangent at A to the circle centre S, for this tangent is perpendicular to SA and A is a point on the auxiliary circle of the conic (see Art. 104). Similarly the conic touches the tangent at B to the circle centre S and the tangents at A, B to the circle centre O.

Consider now the other intersections of the two given circles, namely Ω, Ω'. The tangents to the circle centre S at Ω, Ω' pass through S since S is the pole of $\Omega\Omega'$ with regard to the circle. They are therefore $S\Omega$, $S\Omega'$. But these are also tangents to the conic, since S is a focus.

In like manner the tangents at Ω, Ω' to the circle centre O are tangents to the conic.

Hence the eight tangents at the four common points of two circles touch a conic. Projecting the circles back into any two conics we obtain the result:

The eight tangents to two conics at their four common points touch a conic.

Reciprocating this theorem we obtain the following:

The eight points of contact of the four common tangents to two conics lie on a conic.

174. Involution range on a conic. Let S be a conic, O any point in its plane. Let P be any point on the conic. Join OP meeting the conic again at P'. The correspondence between P and P' is one-one and algebraic, $\therefore$ the ranges $[P]^2$, $[P']^2$ are homographic. Also in this construction P and P' may be interchanged; hence P, P' correspond doubly. The rays through O therefore determine an involution range on the conic, of which the double points are clearly the points of contact of tangents from O to the conic.

Conversely let there be an involution range on the conic, of which (P, P'), (Q, Q') are two pairs of mates. Let PP', QQ' meet at O. Compare the given involution with the one obtained from the intersections of rays through O with the conic. These two involutions have two common pairs of mates, namely (P, P'), (Q, Q'): they are therefore identical by Art. 149. Hence joins of mates of an involution range on a conic pass through a fixed point which is called the *centre* of the involution on the conic.

Also by the property of the cross-axis of two projective ranges on a conic (Art. 111) the meets of cross-joins $(PQ, P'Q')$ and $(PQ', P'Q)$ lie on a fixed line passing through the double

points. This is called the axis of the involution and is clearly the polar of the centre of involution.

By means of this property the double points of an involution range on a conic can be immediately constructed as soon as two pairs of mates (P, P'), (Q, Q') are given. For the axis of involution is the join of $(PQ, P'Q')$ and $(PQ', P'Q)$ and this meets the conic at the double points required.

175. Construction of double rays of an involution flat pencil. The above property of the involution range on a conic may be used to construct the double rays of a flat pencil in involution of which two pairs of mates (p, p'), (q, q') are given. Describe any circle (or conic) through the vertex of the pencil. The involution pencil determines on the circle an involution range, in which (P, P'), (Q, Q') are pairs of mates; P, P', Q, Q' being the points where p, p', q, q' respectively meet the circle. Determine the double points of this involution range on the circle as above and join them to the vertex of the flat pencil. The joins give the double rays required.

176. Involution of tangents to a conic. By reciprocating the theorems of Art. 174 we obtain the results: mates in an involution of tangents to a conic meet on a fixed line, which we call the involution axis. Also joins of cross-meets $(pq, p'q')$, $(pq', p'q)$ pass through a fixed point, which we call the involution centre. In this, as in other theorems on reciprocation, the reader will find it a useful exercise to construct the proof of the reciprocal theorem from that of the given theorem, by reciprocating each step.

The double tangents of the involution are clearly the tangents at the points where the involution axis meets the conic. Also, as in the case of the range, the centre and axis of involution are pole and polar with regard to the conic.

From two pairs of mates (pp'), (qq') the centre and axis of involution are at once constructed and either of these will give the double tangents.

177. The Frégier point. An involution flat pencil whose vertex is on the conic determines an involution of points on the conic. In particular, if the involution pencil be rectangular, we reach the following theorem. If O be any point on a conic, OP, OP' two perpendicular chords, meeting the conic at P, P' respectively, PP' passes through a fixed point F. Taking P coincident with O, OP, OP' are the tangent and normal at O and

PP' coincides with the normal at O. The fixed point F therefore lies on the normal at O. The point is called the *Frégier point* from its discoverer.

If the conic be a rectangular hyperbola and OP, OP' be drawn parallel to its asymptotes, PP', and therefore the Frégier point, is at infinity. In any other position, therefore, PP' is parallel to the normal at O. Thus if on any chord PP' of a rectangular hyperbola as diameter, a circle be constructed meeting the curve at O, O' the normals at O, O' are parallel to PP'.

178. Involution axial pencil. The properties of an involution of planes through an axis are closely similar to those of an involution of coplanar rays through a point. Such an involution determines corresponding involutions, of points on any straight line which cuts it, of rays on any plane which cuts it. By constructing the double elements of either of these the double planes of the axial pencil may be found. As before, if two involutions of planes have the same axis they have one pair of common mates, which is always real unless the two given involutions have two pairs of real double planes which are separated by one another.

The relation between the dihedral angles of six planes in involution is found by taking a section by a plane perpendicular axis. The angles of the flat pencil so found measure the dihedral angles of the axial pencil. These are therefore connected by the formulae of Art. 159.

Also planes at right angles form an involution of which the double planes pass through the circular points at infinity in the plane perpendicular to the axis.

Precisely as at the end of Art. 162 we can show that every involution of planes through an axis has one pair of perpendicular elements.

EXAMPLES X.

1. Prove that any involution pencil can be projected into a rectangular involution.

2. Show that the tangents at the points of an involution on a conic form an involution of tangents having the same axis and centre as the given involution of points.

3. If a set of circles be drawn, each passing through a pair of mates of an involution on a straight line, the radical axes of these circles taken in pairs all pass through one fixed point.

4. If two straight lines meet three circles in three pairs of points of an involution, the three circles have, in general, a common radical axis. Discuss the case of exception.

5. If (A, B) be the double elements of an involution in which (P, P') $(Q. Q')$ are pairs of mates, prove that (A, B) are mates in the involutions determined by the mates (P, Q'), (P', Q) or (P, Q), (P', Q').

6. If (A_1, A_2), (B_1, B_2), (C_1, C_2) be three pairs of points of an involution on a straight line, show that

$$\frac{C_1A_1}{C_2A_2}B_1B_2 + \frac{C_1B_1}{C_2B_2}B_2A_1 + \frac{C_1B_2}{C_2B_1}A_1B_1 = 0.$$

7. Show that if x, x' be the distances of two mates in an involution on a straight line from a fixed origin in the line, then

$$Axx' + B(x + x') + C = 0,$$

A, B, C being constants.

8. Show that two concentric conics have one pair of common conjugate diameters and that these are always real if one of the conics is an ellipse.

The tangent at P to a conic meets a concentric conic at Q, R. Show how to find P so that QR shall be bisected at P.

9. Show that the four points where the tangents from Ω, Ω' touch a conic lie on the orthoptic circle.

10. Any point P on a conic and the pole of the normal at P are conjugate points with regard to the orthoptic circle of the conic.

11. A and B are two fixed points on a conic and PT, PT' the tangents from a variable point P. Prove that if the cross-ratio of the pencil $P(ABTT')$ is constant the locus of P consists of two other conics having double contact with the given conic at A and B.

[Project A, B into Ω, Ω' and use Art. 166.]

12. Show that if P and Q be the two distinct points of contact of a common tangent to two conics which touch at L, and R be the point at which the tangent at L to the conics meets PQ, then $RPUQ$ is a harmonic range, U being the point where a common chord of the two conics meets the common tangent.

[Project the two conics into circles.]

13. Given two points A, B on a conic, find two other points P, Q on the conic such that A and B shall lie on a circle of which PQ is a diameter.

[P, Q are the intersections with the conic of the line joining the Frégier points corresponding to A and B.]

14. If (A, A') be a fixed pair of mates in an involution on a straight line, (P, P') any other pair of mates, prove that $\left(\dfrac{AP.AP'}{A'P.A'P'}\right) = \text{constant}$, and find the value of this constant in terms of the distances of A, A' from the centre of the involution.

15. Show that any line through the cross-centre of two projective pencils meets the two pencils in an involution. Find the mate of the cross-centre: find also the centre of this involution when the given pencils are pencils of parallel rays.

16. From the first result of Ex. 15 prove that the three pairs of opposite sides of a complete quadrangle meet any straight line in three pairs of an involution.

17. The sides BC, CA, AB of a triangle ABC meet a straight line at P, Q, R. If P', Q', R' are mates of P, Q, R in an involution, prove that $P'A$, $Q'B$, $R'C$ are concurrent.

18. If through the vertices of one triangle lines a_1, b_1, c_1 be drawn parallel to the sides of another triangle, and through the vertices of the latter triangle lines a_2, b_2, c_2 be drawn parallel to the sides of the first triangle, prove that if a_1, b_1, c_1 are concurrent, so are a_2, b_2, c_2.

19. A, B are fixed points on a fixed tangent a to a conic s. P, P' are harmonically conjugate with regard to A, B. If p, p' be the tangents from P, P' to s, show that pp' lies on a fixed straight line.

20. If a figure be inverted with regard to any origin, show that an involution on a circle inverts into an involution on the corresponding circle.

21. Show that in any conic if G, G' be the points where the normal at P meet the axes, F the Frégier point corresponding to P, then P, F are harmonically conjugate with regard to G, G'.

22. Show that in a parabola the locus of the Frégier point is another parabola, equal to the given one.

23. A system of conics through four points A, B, C, D determines an involution on any conic through two of them, A and B.

In particular a system of coaxial circles determines an involution on any given circle.

24. Three chords A_1A_2, B_1B_2, C_1C_2 of a circle are concurrent. If O be the centre of the circle, prove the relation

$$\sin \tfrac{1}{2} B_1OC_2 \,.\, \sin \tfrac{1}{2} C_1OA_1 \,.\, \sin \tfrac{1}{2} A_2OB_2 = - \sin \tfrac{1}{2} B_2OC_1 \,.\, \sin \tfrac{1}{2} C_2OA_2 \,.\, \sin \tfrac{1}{2} A_1OB_1$$

and similar relations.

25. Show that if P, P' be a variable pair of points on a line, symmetrically situated with regard to a point O in the line, P, P' are mates in an involution: and find the double points of this involution.

26. P is a point on a fixed straight line u, which meets a conic s at A, B. The tangents from P to s meet the tangent t to s parallel to the tangent at A or B at P_1, P_2. If C be any fixed point on t, prove that

$$CP_1 + CP_2 = \text{constant}.$$

27. Show that any two concentric projective pencils in a plane can always be projected into directly equal pencils.

28. Show, by considering the circle as the product of two directly equal pencils and applying the construction of Art. 116 for its asymptotes, that each of the circular lines through a point may be looked upon as making any given angle with itself.

29. Show analytically that the circular lines are parallel to the lines $y = \pm ix$ and verify that they make the same angles $\pm \tan^{-1} i$ with every straight line in the plane.

30. Discuss the form assumed by the anharmonic property of four fixed points and one variable point on a conic, when two of the fixed points are the circular points.

31. Prove that if t be the product of conjugate pencils with regard to a conic s through two points A, B, then AB has the same pole with regard to s and t.

32. Show that, if t be the product of conjugate ranges with regard to a conic s on two straight lines a, b, then the point U where a, b meet has the same polar with regard to s and t.

Prove also that if a tangent to t meet s at P, Q the lines UP, UQ are harmonically conjugate with regard to a, b.

33. Deduce from Ex. 32 that if mates in an involution pencil meet a conic at P, P', then PP' touches a fixed conic.

34. Prove that chords of a conic s which subtend a right angle at a fixed point O envelop a conic of which O is a focus and the polar of O with regard to s is a directrix.

35. Show that if a simple quadrilateral exist which is inscribed in a conic s and circumscribed to a conic s', there exist an infinite number of such simple quadrilaterals and they have the same intersection of diagonals.

36. Prove that if OA, OB be two lines intersecting at O the cross-ratio $O\{A\Omega B\Omega'\} = e^{2i\theta}$, where $\theta =$ angle AOB.

CHAPTER XI.

THE HOMOGRAPHIC PLANE FORMS OF THE SECOND ORDER.

179. Incident forms. In what follows two unlike homographic forms will be called *incident* if each element of one is incident with the corresponding element of the other. Thus a range and a pencil are incident if each ray of the latter passes through the corresponding point of the former. The pencil formed by the tangents to a conic is incident with the range formed by their points of contact. A flat pencil through a point O of a conic and the range of the second order determined by it on the conic are incident.

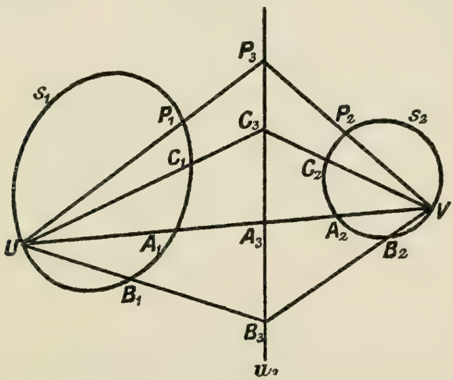


Fig. 47.

The name *perspective* is used by Reye to denote this relation, but this leads to some confusion, for the term perspective is applied to ranges which are sections of the same flat pencil and to pencils

which are projections of the same range, and such forms are certainly not incident.

180. Construction of coplanar homographic forms of the second order. Let two ranges of the second order on two conics s_1 and s_2 be given by two corresponding triads $A_1B_1C_1$, $A_2B_2C_2$ (Fig. 47). Corresponding points of the two ranges may be constructed as follows.

Join A_1A_2 meeting s_1 at U and s_2 at V. Let UB_1, VB_2 meet at B_3 and UC_1, VC_2 meet at C_3. Join $B_3C_3 = u_3$ meeting

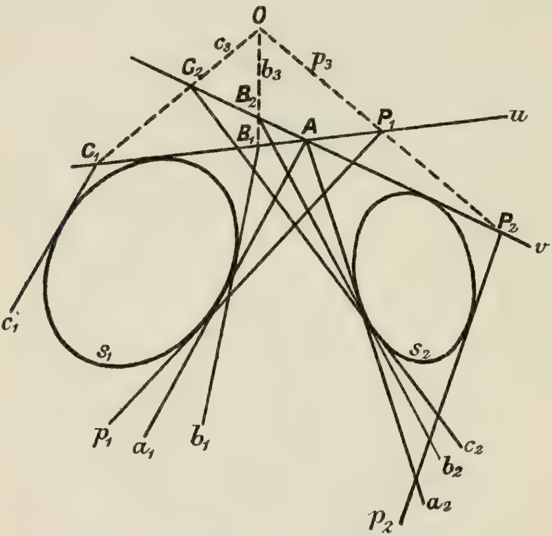


Fig. 48.

UV at A_3. Then if P_1 be any point on s_1 and UP_1 meets u_3 at P_3 and VP_3 meets s_2 at P_2, the ranges of the second order $[P_1]^2$, $[P_2]^2$ are homographic and they have $A_1B_1C_1$, $A_2B_2C_2$ for corresponding triads. They are therefore the ranges required.

If one of the ranges, say $A_2B_2C_2$, is of the first order, a similar construction holds, but this time V may be taken any point on A_1A_2.

Similarly if two pencils of second order about conics s_1, s_2 be given by the corresponding triads $a_1b_1c_1$, $a_2b_2c_2$ (Fig. 48),

then from A $(=a_1a_2)$ draw the two tangents u, v to s_1, s_2. Let $ub_1=B_1$, $uc_1=C_1$, $vb_2=B_2$, $vc_2=C_2$. Let $B_1B_2=b_3$, $C_1C_2=c_3$ and let O be their intersection; let $OA=a_3$. Then if p_1 be any tangent to s_1 meeting u at P_1, and if OP_1 be joined to meet v at P_2 and p_2 be drawn from P_2 to touch s_2, the pencils of the second order $[p_1]^2$, $[p_2]^2$ are homographic and have $a_1b_1c_1$, $a_2b_2c_2$ for corresponding triads.

A similar construction holds if the pencil $[p_2]$ is of the first order, only now v may be taken any line through A and p_2 is joined to the vertex of the pencil $[p_2]$ instead of being drawn tangent to a conic.

If the given forms are unlike, say a range and a pencil of second order, we can correlate as above the given range of the second order with the range formed by the points of contact of the given pencil of second order. In this way the two original forms are geometrically connected.

181. Number of self-corresponding elements of homographic forms of first and second order, not on the same base. Clearly a range of the first order and one of

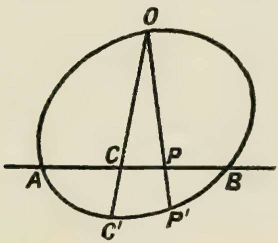


Fig. 49.

the second order cannot have more than two self-corresponding elements since a straight line meets a conic in two points only. They *may* have two self-corresponding elements, for if we take a flat pencil whose vertex is on a conic, the ranges determined by this pencil on the conic and on any straight line have the intersections of the straight line and conic for self-corresponding points.

Conversely if such ranges have two self-corresponding points, say A, B, the lines joining their corresponding points pass through a vertex O lying on the base of the range of second order. For

let C, C' be corresponding points on the straight line and conic respectively (Fig. 49). Join CC' meeting the conic at O. Then if P, P' are on a line through O, the ranges $[P]$, $[P']^2$ are projective. But they are determined by the same triads ABC, ABC' as the original ranges. They are therefore identical with these ranges.

In like manner a pencil of the first order and one of the second order may have two self-corresponding elements, namely the tangents from the vertex of the first pencil to the conic which is the base of the second pencil, but they cannot have more.

Also, when they do have two self-corresponding lines, we

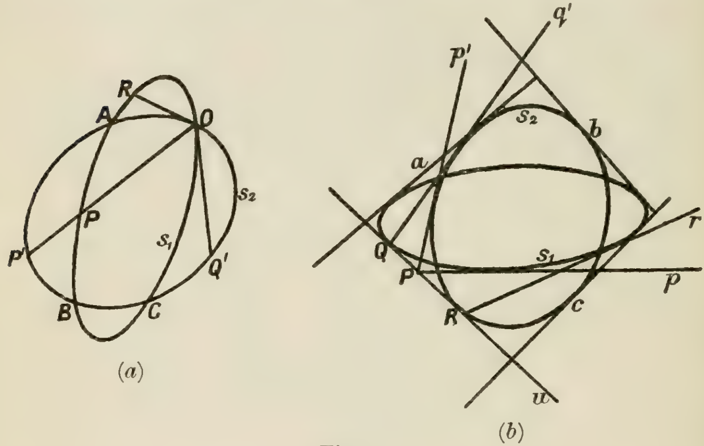


Fig. 50.

can show, by reasoning similar to that used for the ranges, that corresponding lines intersect on a tangent to the conic which is the base of the pencil of second order.

182. Two homographic forms of the second order cannot have more than three self-corresponding elements. Two ranges of the second order have their bases s_1 and s_2 intersecting in four points, but they cannot have more than three self-corresponding points.

In the first place we will show that they may have three self-corresponding points. For let O, A, B, C (Fig. 50 a) be the

four intersections of the conics s_1, s_2; through O draw any ray to meet s_1 at P and s_2 at P', then the ranges $[P]^2$, $[P']^2$ are homographic and they have the points A, B, C self-corresponding.

In the second place two such ranges cannot have more than three self-corresponding points. For let A, B, C be self-corresponding points. Then the self-corresponding triad ABC determines the two ranges completely. Hence the ranges described on the conics by a ray OPP' through O are the only possible ranges satisfying the given conditions. Now such ranges cannot have a fourth self-corresponding point. For such a point could only be at O, and, unless the conics touch at O, O cannot be a self-corresponding point, for if P be at O, P' is at Q' where the tangent at O to s_1 meets s_2, and if P' be at O, P is at R where the tangent at O to s_2 meets s_1. Therefore O is not in general self-corresponding.

If the conics touch at O, O is indeed self-corresponding, but the conics, having already two coincident intersections at O, can have only two other distinct intersections; so that in any case there are not more than three self-corresponding points.

In like manner two homographic pencils of tangents to two conics s_1, s_2 (Fig. 50 b) can have at most three self-corresponding elements, namely three of the common tangents to s_1, s_2. For, if a, b, c be these common tangents, they determine, as in the case of ranges, the relation between corresponding tangents of the pencils, namely, that two such corresponding tangents p, p' meet at P on the fourth common tangent u to s_1, s_2. But u is not self-corresponding; for if p is taken coincident with u, P is at the point of contact Q of u with s_1 and the tangent q' from Q to s_2 is not coincident with u, unless s_1 and s_2 touch u at the same point. But in this case two of the common tangents coincide with u and there are only two others remaining.

183. If a form of the first order has more than three elements incident with their corresponding elements of a homographic form of the second order, the two forms are altogether incident. In the case of two forms of the first order, if more than two elements are incident, the forms are incident. Thus, if three rays of a flat pencil, a_1, b_1, c_1, pass through the corresponding points A_2, B_2, C_2 of a homographic range on a straight line u, the range determined by the pencil upon u has three self-corresponding points with the original range and so coincides with it.

In the case of forms of the second order, however, this no

longer applies, because the range determined by a pencil of the second order upon any straight line is not homographic with the pencil, unless the straight line happen to be a tangent to the conic which is the base of the pencil. The student can easily convince himself of this by reversing the process, when he will find that, although to each tangent to the conic corresponds only *one* point of the straight line, to each point of the straight line correspond *two* tangents to the conic. The correspondence is therefore not one-one.

Consider a range of the second order $[P]^2$ on a conic s and a homographic pencil $[p']$ of the first order whose vertex is U. Take a vertex O on s and join $OP = p$. The pencil $[p]$ is of the first order and $[p] \overline{\wedge} [P]^2 \overline{\wedge} [p']$. The locus of $Q = pp'$ is a conic t passing through O. The conics s, t have therefore three other intersections besides O, of which at least one is real, since by Art. 125 imaginary intersections occur in pairs and O is already one real intersection. But at an intersection of s, t the ray p' passes through its corresponding point P, and, conversely, if p' pass through P, P is an intersection of s, t. Hence there are three pairs of corresponding elements incident, of which one pair is always real.

This holds even if O be on its corresponding ray, for then, if P be at O, OP is tangent to the conic t (since it corresponds to the join of the vertices). But OP is also tangent to the conic s. Thus s, t touch at O and have only two other intersections.

If then there were a fourth pair of corresponding elements incident, the conics s, t would have five points common and would coincide; every pair of corresponding elements would be incident, and, since the pencil and the range are homographic, this can be the case only if the vertex of the pencil lies on the conic which is the base of the range. Otherwise to each ray of the pencil would correspond the two points in which it cuts the conic.

Reciprocating this theorem we obtain the corresponding theorem for a range of the first order and a pencil of the second. The proof is precisely similar to the one above if we interchange terms according to the rules given in Art. 65. The result runs:

If a pencil of the second order and a range of the first order be homographic, then, in general, three pairs of corresponding elements are incident, of which at least one pair are real. If more than three pairs are incident, the two forms are altogether incident and the base of the range touches the base of the pencil.

184. **The product of two homographic pencils of the first and of the second order respectively is a cubic.** If we form the product of two coplanar homographic pencils $[p]$, $[p']^2$, we obtain a curve in the plane. Draw any straight line u in the plane. This meets $[p]$ in a homographic range of the first order $[P]$. Then, if $Q=pp'$, a point P of u is on the locus of Q if it lies on its corresponding line p'. By the last Article there are three such points P on every line u, of which one at least is real. The locus is therefore one of the third degree, or, as it is called shortly, a cubic.

185. **The vertex of the pencil of the first order is a double point on the cubic.** Let O be the vertex of the pencil of the first order, s the conic which is the base of the pencil of the second order. Let u', v' be the two tangents from O to the conic, u, v their corresponding rays through O.

Then O appears *twice* on the locus, once as uu' and once as vv'. Also corresponding to these two interpretations of O there is a different tangent to the curve. For if p' approaches u', p approaches u and the point $Q=pp'$ approaches O so that OQ approaches u. u is therefore one tangent to the curve through O. Similarly if p' approaches v', Q approaches O as OQ approaches v. So that v is another tangent to the curve at O. The curve has two branches which intersect at O.

Such a point O is called a double point on the curve and every line through O is considered as meeting the curve in two coincident points at O. The only case of a double point which we have met with hitherto is that of the degenerate conic or line-pair, where the intersection of the two lines is a double point.

This we can verify by noting that a ray through O meets the locus again at one point only, as it should, to wit, at the point where it is met by the corresponding ray of the pencil of second order. There are two exceptions, however, namely the rays u, v. These meet the curve in three coincident points at O and are known as the proper tangents to the curve at O.

It may be shown that, in order that a cubic may have a double point, a certain condition must be satisfied. Hence the cubic of the present Article is not of the most general type.

186. **Construction of directions of the asymptotes of this cubic.** The points where the cubic meets the line at infinity i^∞ may be constructed as follows. Draw a tangent a to s (Fig. 51) from O. Let a' be the tangent corresponding to

the ray a through O. Let c' be the tangent parallel to a, c its corresponding ray. Let b, b' be any other pair of corresponding lines. Let a, b, c meet i^∞ at A^∞, B^∞, C^∞ and a', b', c' meet a at A', B', C'^∞ ($C'^\infty = A^\infty$). If p, p' be any pair of corresponding lines (not shown in Fig. 51) meeting i^∞, a at P^∞, P' respectively the ranges $[P^\infty]$, $[P']$ are projective, and $A^\infty B^\infty C^\infty$; $A'B'C'^\infty$ are two corresponding triads. The parallel $P^\infty P'$ through P' to p therefore envelops a parabola which touches a and i^∞ at A', C^∞

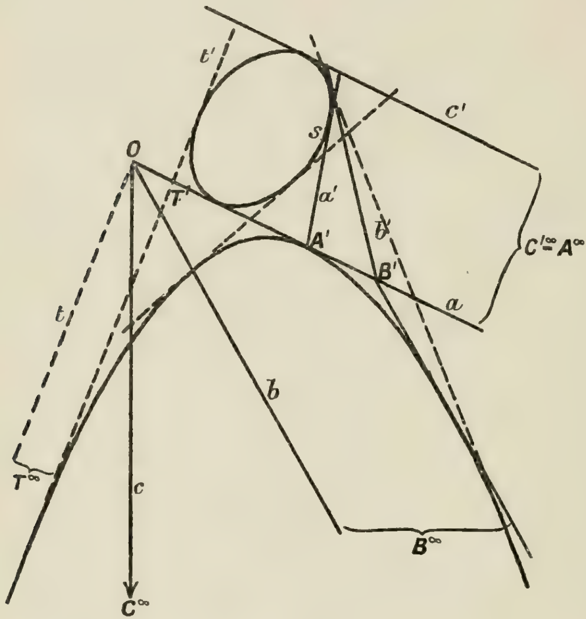


Fig. 51.

respectively, these being the correspondents to ai^∞ ($= C'^\infty = A^\infty$). This parabola also touches $B'B^\infty$, i.e. the parallel through B' to b. We are therefore given one tangent $B'B^\infty$, another tangent a and its point of contact A', and the direction of the axis c. The parabola can then be drawn by Brianchon's Theorem. The three common tangents to the parabola and to s, other than a, then give the directions of the three asymptotes of the cubic. For let t' be any one of these common tangents, meeting a at T', and if t be the

corresponding ray, meeting the line at infinity at T^∞, $T'T^\infty$ is a tangent to the parabola and therefore in the same straight line as t'. Thus tt' is T^∞, a point at infinity on the cubic.

187. The product of two homographic ranges of the first and of the second order respectively is a curve of the third class with a double tangent. If we form the product of homographic ranges $[P]$, $[P']^2$ on a line x and a conic s respectively, and take any point U in the plane, then if $p = UP$, the pencil $[p]$ and the range $[P']^2$ are homographic. They have three incident pairs which correspond to those cases where U, P, P' are collinear, i.e. where UP is tangent to the envelope of PP'. The envelope is thus of the third class since from any point three tangents can be drawn to it.

Let the two points where x meets s be T', V'. Let the corresponding points on x be T, V. Then x occurs twice as a tangent to the envelope. It is therefore a double tangent and touches the envelope at T, V.

188. Degenerate cases of the above. If the two homographic pencils of Art. 184 have one of the tangents from O to s as a self-corresponding ray, the whole of this ray forms part of the locus. The locus of the third degree breaks up therefore into a straight line and a conic. If the two tangents from O are self-corresponding rays, the whole of each of these rays is part of the locus and the cubic breaks up into three straight lines, namely the two tangents from O and a third tangent to s (Art. 181).

In like manner if the ranges of Art. 187 have a self-corresponding point, that point is an isolated part of the envelope. The envelope breaks up into this point and a curve of the second class, i.e. a conic. If the ranges have two self-corresponding points, the envelope breaks up into three points, one of which is on the base of the range of second order by Art. 181.

189. Two homographic unlike forms of the second order have four pairs of corresponding elements incident; and if they have more than four, they are altogether incident. Let $[p]^2$, $[P]^2$ (Fig. 52) be a homographic pencil and range respectively, of the second order. On the base s_2 of the range $[P]^2$ take a point O. Then the pencils $O[P]$, $[p]^2$ are homographic. Let Q be the meet of p, OP. The locus of Q is a cubic of which O is a double point. But a conic and a cubic are known from analytical considerations to have six intersections. Hence the locus of Q meets s_2 at six points. Of these O counts

as two, since O is a double point. There are accordingly four others A, B, C, D. Corresponding to each such intersection we have a point P on its corresponding line (since P and Q are then coincident).

Or, if it be desired to avoid the above appeal to analytical considerations, we may proceed as follows. The locus of Q must meet s_2 at some point A (real or imaginary). Through such a point A the corresponding line a passes. Take for O the point where a meets s_2 again. The pencils $O[P]$, $[p]^2$ have now a for a self-corresponding ray. The locus of Q now reduces to a conic

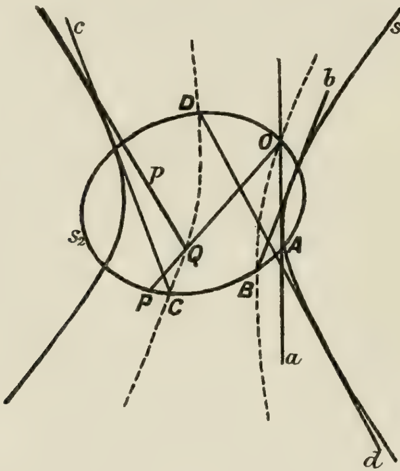


Fig. 52.

v through O (a being irrelevant). This conic v (shown by the dotted line in Fig. 52) cuts s_2 at three other points B, C, D, which are incident with their corresponding lines. O is not incident with its corresponding line, unless v touches s_2 at O; for if p passes through O, Q is at O and P is where OQ, i.e. the tangent at O to v, cuts s_2, that is, P is not at O if v, s_2 do not touch at O. But if they do touch at O, then v, s_2 have only two other points of intersection and there are still only four points on their corresponding lines.

Hence if there be a fifth point E through which passes its corresponding line e, the conics v, s_2 have five points in common

and coincide entirely. Thus every point lies on its corresponding line and the two given forms are incident.

190. Product of cobasal homographic forms of the second order. Let $[P_1]^2$, $[P_2]^2$ be two homographic ranges of the second order on the same conic s. Let A_1, A_2 be a given pair of corresponding points of these ranges and ST the cross-axis (Fig. 53 *a*). Then A_1P_2, A_2P_1 meet at U on the cross-axis. Project S, T into the circular points at infinity Ω, Ω'. s projects into a circle s' (Fig. 53*b*), U into a point U'^{∞}. Therefore $A_1'P_2'$, $A_2'P_1'$ are parallel, and the arcs $A_1'P_1'$, $A_2'P_2'$ are directly equal. The ranges $[P_1']^2$, $[P_2']^2$ on the circle are thus determined by

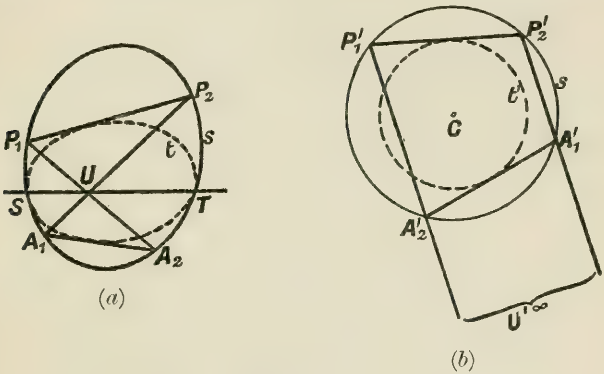


Fig. 53.

directly equal flat pencils whose vertex is on the circle. The arc $P_1'P_2'$ subtends a fixed angle at the circumference and therefore at the centre. Hence the chord $P_1'P_2'$ touches a fixed concentric circle t'.

Now two concentric circles touch one another at Ω, Ω'. For if C be their common centre the tangents at Ω, Ω' are $C\Omega$, $C\Omega'$ and are the same for both. Projecting back into the original figure, P_1P_2 touches a fixed conic t which has double contact with the original conic at S and T.

Reciprocating this theorem, we see that the product of two homographic pencils of tangents to the same conic s is a conic having double contact with the original conic along the two tangents to this conic from the cross-centre of the pencils.

In the special case where these homographic forms are in involution, the above envelope and locus degenerate into a point and a straight line respectively, the point appearing as the intersection of the two components of a line-pair and the line as the join of the components of a point-pair. The line-pair in the first case is the pair of tangents from the centre of involution to the conic and the point-pair in the second case is the pair of points at which the axis of involution meets the conic.

The above theorems are of considerable importance and a number of interesting particular deductions flow from them. In particular let there be two ranges on the line at infinity i^∞ defined by the intersections with i^∞ of two directly equal pencils in which corresponding rays make an angle α with one another, and let Q^∞, Q'^∞ be corresponding points of these ranges. Since i^∞ touches every parabola, the tangents from Q^∞, Q'^∞ to a parabola define two homographic pencils of tangents to the parabola. Their product is a conic meeting i^∞ at the two points corresponding to the point of contact of i^∞ with the parabola. Hence we have the theorem: the locus of intersections of two tangents to a parabola which make an angle α (other than right) with one another is a hyperbola whose asymptotes make an angle α with the axis of the parabola.

191. Homographic involutions. We may treat a pair of mates in an involution as a single entity and establish a one-one algebraic correspondence between the pairs of one involution and the pairs of another; such correspondence does not establish any one-one relation between the individual mates, but only between the pairs as a whole. Two involutions correlated in this way will be called homographic.

We will first show how to derive two homographic involution ranges on the same conic s one from the other. Let (P_1, P_1'), (P_2, P_2') (Fig. 54) be two corresponding pairs; let O_1, O_2 be the corresponding involution centres and p_1, p_2 the rays through O_1, O_2 determining the corresponding pairs on s. Then by hypothesis the rays p_1, p_2 are connected by a one-one algebraic correspondence. The pencils $[p_1]$, $[p_2]$ are homographic: if $Q = p_1 p_2$ the locus of Q is a conic which meets s at four points A, B, C, D.

When Q is at A, one point of a pair (P_1, P_1') of one involution coincides with one point of the corresponding pair (P_2, P_2') of the other involution, though it should be noted carefully that the pairs as a whole do not in general coincide. Such a point as A

will be spoken of as a self-corresponding point of the homographic involutions. Since there are four such points A, B, C, D two homographic involutions of points on the same conic have four self-corresponding points.

But any two involutions, of any type, may always be uniquely correlated with two involution ranges on the same conic, e.g. two involution ranges on different straight lines may be projected from a vertex as two concentric involution pencils and then cut by a conic through the vertex; and two involution ranges on different conics may be projected from vertices on the conics into

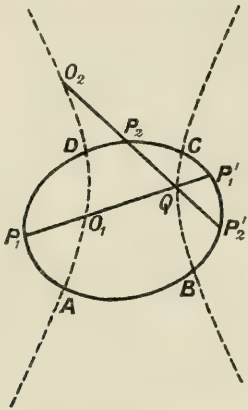


Fig. 54.

two non-concentric involution flat pencils and then cut by a conic through the two vertices; also two involution pencils of tangents to two conics are correlated with the involution ranges formed by their points of contact.

Hence the above method can ultimately be used to derive geometrically any two homographic involutions one from the other.

Also we have the general theorem that two cobasal homographic involutions have four self-corresponding elements.

192. Case where double elements correspond. In general, in two homographic involutions, no homographic relation exists between the individual components of the pairs. In the case where the double elements correspond, however, such a relation can be shown to exist.

We notice first that, since, from the last Article, the relation between two homographic involutions is determined by a relation between two homographic simple forms, which latter is itself determined by two corresponding triads, two corresponding triads of pairs can be arbitrarily assumed and completely determine the relation between two homographic involutions.

Suppose now that A_1, B_1, the double elements of one involution, and A_2, B_2, the double elements of another involution, correspond. Let the pair (C_1, C_1') correspond to the pair (C_2, C_2'). Then by the property of involutions that any pair are harmonically conjugate with regard to the double elements, we have

$$\{A_1C_1B_1C_1'\} = \{A_2C_2B_2C_2'\},$$

and the sets of four elements $A_1C_1B_1C_1'$, $A_2C_2B_2C_2'$ can be brought into homographic correspondence. Now the homographic correspondence thus defined will transform the involution (P_1, P_1') into a homographic involution (P_3, P_3') cobasal with (P_2, P_2') and homographic with it. But the cobasal homographic involutions (P_3, P_3'), (P_2, P_2') have three self-corresponding pairs, namely the double elements A_2, B_2 and the pair (C_2, C_2'). Therefore they must be identical. Hence this homographic correspondence connects P_1 with P_2 and P_1' with P_2', i.e. there is a homographic correspondence between the individual components of the pairs. Similarly a homographic correspondence exists which connects P_1 with P_2' and P_1' with P_2.

193. Product of two homographic involutions of the first order. If we form the product of two homographic involution pencils of vertices O_1, O_2, that is, find the intersections p_1p_2, p_1p_2', $p_1'p_2$, $p_1'p_2'$ where (p_1, p_1'), (p_2, p_2') are two corresponding pairs, these intersections lie on a certain locus.

As in Art. 115 we proceed to find the intersections of this locus with any straight line x. The two involution pencils determine on x two collinear homographic involutions of which (P_1, P_1') (P_2, P_2') are corresponding pairs, where $P_1 = p_1x$, etc. If either of the points P_1, P_1' coincide with either of the points P_2, P_2', the point of coincidence lies on a ray of each of two corresponding pairs of the given homographic involution pencils, that is, it lies on their product.

But, by the last Article, there are four such self-corresponding points of the collinear homographic involution ranges (P_1, P_1') (P_2, P_2'). The locus therefore meets any straight line in four points, that is, it is a curve of the fourth degree.

Also the vertices O_1, O_2 are double points on this curve. For

let (u_1, u_1') be the pair of the pencil vertex O_1 corresponding to the pair of the pencil vertex O_2 of which O_2O_1 is a component. As the ray p_2 approaches O_2O_1, (p_1, p_1') approach (u_1, u_1') and two points on the locus coincide at O_1, moving ultimately along u_1, u_1'. O_1 is thus a double point, (u_1, u_1') being the proper tangents at O_1. In like manner O_2 is a double point and, if (v_2, v_2') is the pair of the pencil through O_2 corresponding to the pair of the pencil through O_1 of which O_1O_2 is a component, then v_2, v_2' are the proper tangents at O_2.

If O_1O_2 happens to be a self-corresponding ray of the pencils, the locus breaks up into O_1O_2 and a cubic curve passing through O_1, O_2. In this case, however, O_1, O_2 are not double points on the cubic, for the two mates to O_1O_2 are now the only tangents at O_1, O_2.

Also if it so happen that the correspondence between the two involutions is of such a nature that individual mates can be brought into one-one correspondence (Art. 192), the locus of the fourth degree breaks up into two conics, these being the products of the two pairs of homographic pencils formed by the individual mates.

Reciprocating the above theorems or proceeding directly in a similar manner, we have the result that the product of two homographic involution ranges of the first order is a curve of the fourth class, to which the bases of the given involutions are double tangents. If the ranges have a self-corresponding point this envelope breaks up into a point and a curve of the third class. If the individual mates can themselves be homographically correlated, it breaks up into two conics.

194. Involution homographic with a simple form. We can extend this method and define in an analogous manner homography between an involution and a simple form. In order to establish the relations between these, we proceed as in Art. 191, and consider a range on a conic s and a homographic involution range on the same conic. The range may be defined by a pencil $[p_1]$ through a vertex O_1 on s and the involution by a homographic pencil $[p_2]$ through a vertex O_2 not on the conic. We obtain the figure if in Fig. 54 we take O_1 on the conic. All points P_1' then coincide with O_1, so that this is really a special case of the last. The product t of the pencils $[p_1]$, $[p_2]$ now cuts s at O_1 and at *three* other points, and it is easy to show, as in Arts. 182, 189, that O_1 is not a self-corresponding point unless t, s touch at O_1. Hence an involution and a homographic simple

form on the same conic have three self-corresponding points; and the result can be extended to cobasal involutions and forms of any type as in Art. 191.

195. Product of an involution and a homographic simple form. Proceeding as in Art. 193 we can show that the product of an involution pencil of vertex O_1 and a homographic simple pencil of vertex O_2 is a cubic having O_1 for a double point and passing through O_2. If O_1O_2 be a self-corresponding element the locus breaks up into a straight line and a conic.

Similarly the product of an involution range on a line u_1 and a simple range on a line u_2 is a curve of the third class having u_1 for a double tangent and u_2 for an ordinary tangent. If u_1u_2 be a self-corresponding point the envelope breaks up into a point and a conic.

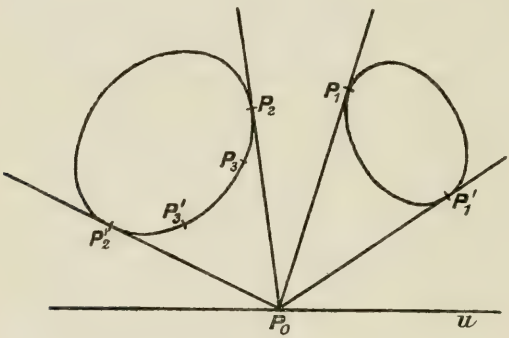


Fig. 55.

196. The product of two homographic pencils of the second order is a curve of the fourth degree. Consider two homographic pencils of tangents about two conics s_1 and s_2. Let u be any straight line in the plane. Take any point P_0 on u (Fig. 55), and draw from P_0 pairs of tangents to s_1 and s_2 touching these conics at P_1, P_1' and P_2, P_2' respectively. Then the involutions (P_1, P_1') (P_2, P_2') are homographic. Let P_3, P_3' be the points of contact of the tangents to s_2 which correspond to the tangents at P_1, P_1' to s_1. Then, owing to the homography between the tangents to s_1 and the tangents to s_2, the pairs (P_3, P_3') form an involution homographic with that formed by the pairs (P_1, P_1'), and therefore with the one formed

by the pairs (P_2, P_2'). Now there are four self-corresponding elements of the homographic cobasal involutions $(P_2, P_2')(P_3, P_3')$. To each of these self-corresponding points corresponds a point P_0 such that through P_0 pass corresponding tangents of the two original pencils. Conversely to every such point P_0 corresponds a self-corresponding point of the involutions (P_2, P_2') (P_3, P_3'). But the points P_0 through which pass corresponding tangents of the original pencils lie on the product of the pencils. Any straight line u therefore meets such a product in four points. Hence the locus is a curve of the fourth degree.

If one of the common tangents is self-corresponding it is part of the locus. The latter then breaks up into this line and a cubic. If a second common tangent is self-corresponding the locus breaks up into two straight lines and a conic. The case where three common tangents are self-corresponding has already been discussed in Art. 182. The locus then breaks up into the four common tangents.

Reciprocating the above we see that the product of two homographic ranges of the second order is an envelope of the fourth class. The student will have no difficulty in tracing the degenerate cases when one, two, or three points are self-corresponding.

EXAMPLES XI.

1. If $[p]$, $[p']^2$ be two homographic pencils of the first and second orders respectively having a self-corresponding ray a which touches the base s of $[p']^2$ at A; and if u be any straight line in the plane, and $up = P$, $ap' = P'$: prove that PP' envelops a conic which touches s at A.

2. If two homographic pencils of the first and second orders respectively have a self-corresponding ray, show how to construct the two intersections of their product with any straight line.

3. If two homographic ranges of the first and second orders respectively have a self-corresponding point, show how to draw the two tangents to their product from any point.

4. From a point O a ray OP is drawn to meet a fixed straight line l at P. If O' be the point of contact of a tangent from O to a fixed circle c and $O'P$ meet the circle again at P', prove that the locus of the intersection of OP and the tangent at P' is a conic.

5. P, Q are two points on a tangent to a conic s. From P, Q tangents p, q are drawn to s, meeting at R. If PQ be of constant length, find the locus of R.

6. Through a fixed point O a ray is drawn to meet a given circle at P. Find the envelope of a straight line through P which makes a constant angle with OP.

7. Show that if two conics have double contact, any tangent to either determines on the other ranges homographic with each other and with the range described by the point of contact.

8. Two conics have double contact at A, B. A chord PQ of one conic slides on the other. Show that the cross-ratio of the four points A, B, P, Q is constant.

9. P is a point on a conic s; from P a tangent is drawn to a conic t which has double contact with s to meet s again at P_1; from P_1 another tangent is drawn to t to meet s again at P_2: and so on. After n operations we reach a point P_n by a chain of tangents. If the chain of tangents slide round t, prove that the ranges $[P]^2, [P_1]^2, [P_2]^2, \dots [P_n]^2$ are all projective and have common self-corresponding points. Prove also that if for one position of P (other than a point of contact of s, t) P_n coincides with P, it will do so for all positions of P.

Deduce that if a polygon of n sides exist which can be inscribed in a conic s and circumscribed to a conic t having double contact with s, an infinite number of such polygons exist.

10. A straight line meets a conic at A, B. On AB points P, Q are taken so that the cross-ratio $\{ABPQ\}$ is constant. From P and Q tangents are drawn to the conic meeting at R. Show that R lies on either of two fixed conics having double contact with the original conic at A and B.

[Project A, B into the circular points.]

11. Prove that a variable circle which cuts two fixed circles at right angles determines on these circles two homographic involutions.

12. s_1, s_2 are two conics, u a fixed tangent to s_1. From a point Q_0 of u tangents Q_0Q_1, Q_0Q_2 are drawn to s_1, s_2. O_1, O_2 are fixed points on s_1, s_2 respectively: O_1Q_1, O_2Q_2 meet at Q. Show that the locus of Q is a cubic having O_2 for a double point and construct the proper tangents to the cubic at O_2.

13. If in two homographic involution pencils of the first order the join of the vertices O, O' is a double ray of each pencil and self-corresponding, prove that the remainder of the product is a conic with regard to which O, O' are conjugate points.

[For the other double ray through O meets its corresponding pair in the points of contact of that pair and so is the polar of O'.]

14. If S be a double point on a cubic, O another fixed point on the cubic, OPQ a ray through O cutting the cubic again at P, Q, show that SP, SQ are mates in an involution.

Hence show that if O, A, B are collinear, and if O, C, D are collinear, the locus of the points of contact of tangents from O to all cubics having a common double point and passing through O, A, B, C, D consists of two straight lines through the double point.

15. Show that the hyperbola which is the locus of intersections of tangents to a parabola making a constant angle a with each other has the same focus and directrix as the original parabola.

[Show that the points of contact of the tangents from Ω, Ω' to the parabola lie on the hyperbola.]

CHAPTER XII.

SYSTEMS OF CONICS.

197. Ranges and pencils of conics. A set of conics passing through four fixed points A, B, C, D are said to form a *pencil of conics.*

Through any fifth point E of the plane there passes one conic of the pencil and one only, since five points determine a conic.

A set of conics touching four fixed lines a, b, c, d are said to form a *range of conics.*

There exists one conic of the range, and one only, which touches any given line e of the plane.

198. Involution determined by a pencil of conics on any straight line. Consider any straight line u. Let P be any point of u. The conic of the pencil through P meets u again at one point P', which is therefore uniquely determined if P be given. Conversely if P' be given P is known. Also, since P and P' determine the same conic of the pencil, when P is taken at P', P' is at P. The ranges $[P]$, $[P']$ on u are therefore connected by a one-one correspondence in which the elements correspond doubly. Hence they form an involution upon u.

The double points S, T of this involution are the points of contact of the conics of the pencil which touch u. This enables us to solve the problem: to draw a conic through four given points A, B, C, D and touching a given straight line u. We see that this problem has in general two solutions which are real only if the involution determined upon u by the pencil of conics is hyperbolic.

Three of the conics of the pencil degenerate into the line-pairs formed by opposite sides of the quadrangle $ABCD$. They are $(AB\,;\,CD)$, $(AC\,;\,DB)$, $(AD\,;\,BC)$. We thus obtain the theorem :

The three pairs of opposite sides of a complete quadrangle

meet any straight line in three pairs of points of an involution (cf. Exs. X. 16).

199. Involution determined by tangents to a range of conics from any point. Proceeding on similar lines, or reciprocating the above theorem, we obtain the result:

If (p, p') be the tangents from a point U to any conic of a range, the rays (p, p') form an involution, of which the double rays s, t are the two tangents at U to the conics of the range which pass through U.

The problem, to draw a conic to touch four given lines and to pass through a given point, has therefore in general two solutions.

Three of the conics of the range degenerate into the point-pairs formed by opposite vertices of the quadrilateral *abcd*. They are $(ab\,;\,cd)$, $(ac\,;\,db)$, $(ad\,;\,bc)$. The tangents from U to these point-pairs are the joins of U to the points of the pair. We have then:

The lines joining any point to the three pairs of opposite vertices of a complete quadrilateral form three pairs of mates of an involution pencil.

From the property of the present Article follows at once the theorem that the orthoptic circles of a range of conics are coaxial.

For consider two such orthoptic circles, intersecting at G, H. The involution of tangents to the system from G has two pairs of rectangular rays and is therefore rectangular. Thus G lies on every orthoptic circle of the system. Similarly for H.

200. Common self-polar triangle of a pencil of conics or of a range of conics. Since by Art. 61 the diagonal triangle of a quadrangle inscribed in a conic is self-polar for the conic, it follows that the diagonal triangle of the quadrangle $ABCD$ which is inscribed in all the conics of the pencil is self-polar with regard to every conic of the pencil. Clearly the vertices of this triangle are the centres (i.e. the intersections of the components) of the three line-pairs of the pencil.

In like manner the diagonal triangle of the quadrilateral *abcd* is self-polar with regard to all the conics of the range defined by a, b, c, d. Its three sides are the lines joining the components of the three point-pairs of the range.

201. Points conjugate with regard to a pencil of conics. The double points S, T of the involution determined by the pencil of conics on any straight line u are harmonically

conjugate with regard to the pair of mates (P, P') in which u is cut by any conic of the pencil. S, T are therefore conjugate points with regard to every conic of the pencil.

Conversely if two points S, T are conjugate with regard to every conic of the pencil, they must be double points of the involution on ST, since they are harmonic conjugates with regard to every pair of mates of this involution.

To every point S of the plane there corresponds one point S' which is conjugate to S with regard to every conic of the pencil, or, as we shall put it for brevity, with regard to the pencil. For draw the conic of the pencil through S. Let s be the tangent to it at S. On s there must be a point S' which is the point of contact of the second conic of the system which touches s. S, S' are therefore the double points of the involution determined by the pencil of conics upon s and so are conjugate points with regard to the pencil. Also there is in general only one such point. For clearly if any line t other than s be drawn through S, it cuts the conic of the pencil through S. Thus S would not be a double point of the involution on t and could have on t no conjugate point with regard to all the conics.

An exception occurs if S be a centre of a line-pair, for every line through S is then a tangent to the line-pair. And indeed in this case S is a vertex of the common self-polar triangle and has an infinite number of points conjugate to itself with regard to the pencil, namely all the points of the opposite side of the triangle.

Since the centres of the three line-pairs are the only points possessing this property, it follows that the pencil of conics have only one common self-polar triangle.

202. Lines conjugate with regard to a range of conics. Proceeding in a precisely similar manner, or reciprocating the results of the last Article, we can show that:

Through any point S of the plane pass two lines s, s' which are conjugate lines with regard to every conic of the range; and they are the double rays of the involution of tangents from S to the conics of the range.

To construct the line conjugate to any given line s with regard to the range, find the point of contact S of s with the conic of the range determined by s. The conjugate line s' is then the tangent at S to the other conic of the range through S.

s' is uniquely determined unless s is the line joining the components of a point-pair of the range. Any point of s may

then be taken as the point of contact of s with the point-pair, and to such a line s correspond an infinite number of lines s', namely the lines through the opposite vertex of the diagonal triangle of *abcd*.

This being the only case of failure, this diagonal triangle is the only common self-polar triangle of the range.

203. The eleven-point conic. We proceed to find the locus of the point S' conjugate to S with regard to a pencil of conics, when S describes any straight line q.

The point S' may be found by a construction other than

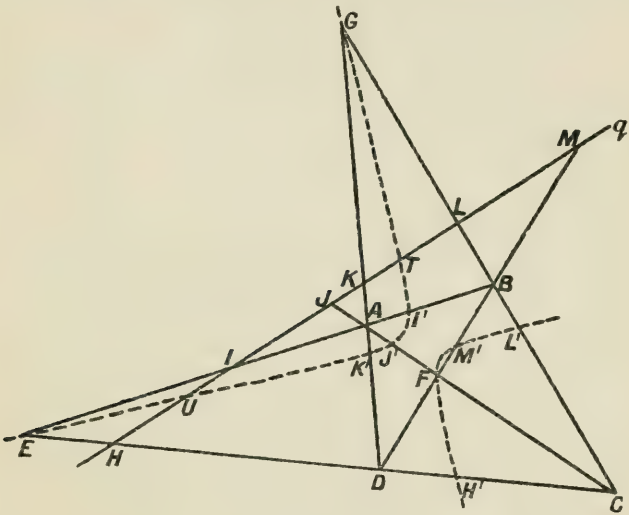


Fig. 56.

the one given in Art. 201. For let v_1, v_2 be any two conics of the pencil, Q_1, Q_2 the poles of q with regard to v_1, v_2 and s_1, s_2 the polars of S with regard to v_1, v_2.

Then s_1 is the locus of points conjugate to S with regard to v_1; s_2 is the locus of points conjugate to S with regard to v_2. Now S' is conjugate to S with regard to both v_1 and v_2. Thus $S' = s_1 s_2$. But as S moves on q, s_1, s_2 describe pencils homographic with the range described by S (Arts. 55, 60). Hence S' describes a conic (Art. 41) passing through Q_1, Q_2.

This conic is known as the *eleven-point conic* of q.

For let EFG (Fig. 56) be the common self-polar triangle of the pencils. Then E is conjugate to the point of q in which q is cut by FG. Therefore E is a point on the locus of S': similarly F, G are points on this locus. Again the two double points T, U of the involution determined by the pencil on q, being conjugate to one another, are on the locus.

Let H, I, J, K, L, M be the points at which q meets the six sides of the quadrangle $ABCD$. Then the harmonic conjugates H', I', J', K', L' M' of H, I, J, K, L, M respectively with regard to the two vertices on the corresponding sides of the quadrangle must lie on the locus. For clearly CD being a chord of all the conics of the pencil, (H, H') are conjugate with regard to all such conics.

The locus of S' thus passes through these eleven points.

Since the eleven-point conic of q passes through the two poles Q_1, Q_2 of q with regard to v_1, v_2; and v_1, v_2 are any conics of the pencil, the eleven-point conic passes through the poles of q with regard to all the conics of the pencil.

It is therefore also the locus of the poles of q with regard to the conics of the pencil.

204. The eleven-line conic. Reciprocating the above theorems we obtain the following results.

The envelope of lines conjugate to the rays of a pencil through a point Q with regard to a range of conics touching a, b, c, d is a conic which touches: (1) the three sides of the diagonal triangle of the quadrilateral $abcd$, (2) the two lines through Q conjugate with regard to the range, i.e. the two tangents at Q to the two conics of the range through Q, (3) the six harmonic conjugates to the rays joining Q to the vertices of the complete quadrilateral $abcd$, taken with regard to the two sides of the quadrilateral through each vertex.

This conic is also the envelope of the polars of Q with regard to the conics of the range.

205. Geometrical constructions for common self-polar triangle of two conics. If two real conics intersect in four real points A, B, C, D, or else lie entirely outside each other, so that they have four real common tangents a, b, c, d, their common self-polar triangle is at once constructed, being the diagonal triangle of the quadrangle $ABCD$ or of the quadrilateral $abcd$.

If, however, two of the points of intersection, say C and D, are conjugate imaginary, the other two A and B being real, the

line CD is a real straight line. The vertex E (Fig. 56) of the self-polar triangle is therefore real and its polar FG with regard to the two conics is also real. But F and G cannot be real: for if F were real, AF would be a real line and its meet C with CD would be a real point, which is against the hypothesis. In this case, then, two vertices of the common self-polar triangle, and also their opposite sides, are imaginary.

If all the points of intersection are imaginary they fall into two conjugate imaginary pairs, say A, B and C, D. Then AB, CD are real lines and their meet E a real point. Also A being conjugate imaginary to B and C conjugate imaginary to D, the line AC is conjugate imaginary to BD by Art. 125, and thus their intersection F is real. Similarly G is real. Hence when all four points of intersection are imaginary, the common self-polar triangle is real.

Proceeding similarly we see that the common self-polar triangle is real when the four common tangents are either all real or all imaginary: it is imaginary when two of the common tangents are real and two imaginary.

Comparing these results with the previous ones we observe that if two conics have only two real intersections they have only two real common tangents; but they may have: (1) four real intersections and four real common tangents, e.g. two conics having their four real intersections on the same branch of each; (2) four real intersections and four imaginary common tangents, e.g. two conics having real intersections on both branches of one of them; (3) four imaginary intersections and four imaginary common tangents, e.g. one conic lying entirely inside another; (4) four imaginary intersections and four real common tangents, e.g. two ellipses lying entirely outside one another.

In case (3) the geometrical construction for the diagonal triangle fails entirely. We can then proceed as follows. Take any two lines p and q. Construct their eleven-point conics as in Art. 203. These two eleven-point conics intersect in four points, namely the vertices E, F, G, of the common self-polar triangle and the point conjugate to pq with regard to both conics. The latter point being always real, we get a new proof that one of the vertices of the self-polar triangle is always real.

206. Construction of conic of pencil through any given point whenever common self-polar triangle is real. Let a pencil of conics be defined by two real conics intersecting at A, B, C, D. Then if A, B, C, D be real the conic

of the pencil through any fifth point is readily constructed by Pascal's Theorem. But if the intersections be not real, construct the common self-polar triangle of the conics by the method of the last Article. Let P be a point through which it is required to draw a conic of the pencil. Then by Art. 59, knowing P and a self-polar triangle, three other points Q, R, S of the required conic are known. Now consider any straight line u which meets the two given conics in two pairs of real points (U, U'), (V, V'). The conics through P, Q, R, S and the conics through A, B, C, D determine two involutions upon u, the latter of which is known from the two pairs (U, U'), (V, V'). The common mates of these involutions therefore give the points in which the required conic of the given pencil meets u. Having now six points on this conic, we can draw it by Pascal's theorem.

The student will find it profitable to work out the corresponding construction for the conic of a range defined by two conics, which touches a given line p of the plane, when four of the common tangents to the given conics are imaginary.

207. Conics having double contact. When two conics have double contact at A and B they define a pencil of conics having double contact with the two given ones at A and B. The pole E of AB is the same for all the conics; and if F, G be any pair of points on the common chord of contact harmonically conjugate with regard to A, B, EFG is a common self-polar triangle of the pencil of conics. There is thus an infinity of common self-polar triangles. The three line-pairs of the system degenerate into the doubled line AB, occurring twice over, and the pair of common tangents EA, EB.

Also such a pencil of conics may be looked upon as forming a range, the four common tangents being coincident in pairs. Such a set of conics partakes of the properties both of the pencil and of the range. Thus to any point there is a conjugate point and to any line a conjugate line, with regard to all the conics of the set.

Hence the locus of poles of any straight line with regard to the conics of the set is a straight line and the polars of a point pass through a point. It is of interest to see how these occur as degenerate cases of the eleven-point and eleven-line conic respectively.

Consider any point Q (Fig. 57) on a straight line q. Let q meet AB at R and let R' be the harmonic conjugate of R with respect to A, B. Since R, R' are conjugate with regard to all conics of the set, and E, R are conjugate with regard to all conics

of the set, R is the pole of ER' ($=q'$) with regard to all the conics of the set. And since q passes through R, q' is the locus of poles of q with regard to the conics of the set. On the other hand consider the ray harmonically conjugate to EQ with regard to EA, EB. Let it meet AB at Q', and let EQ meet AB at T. Then $Q'ET$ is a self-polar triangle for all the conics of the set, or Q' is conjugate to Q with regard to the set of conics.

The eleven-point conic corresponding to q therefore breaks up into two straight lines, of which one AB is the locus of points conjugate to points of q, and the other ER' is the locus of poles. Thus these two loci, which are the same in the general case, are

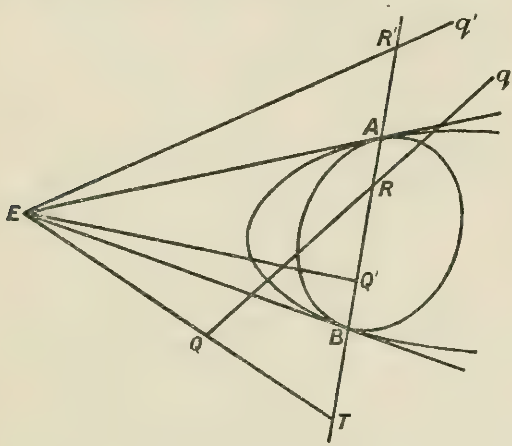


Fig. 57.

now separated. In like manner the eleven-line conic corresponding to Q breaks up into the point E which is the envelope of lines conjugate to lines through Q, and the point Q' which is the envelope of polars of Q with regard to the set of conics.

208. Construction of conics through three points and touching two lines. Let it be required to construct a conic to pass through three points A, B, C and to touch two lines p, q (Fig. 58). Let the conic required touch p, q at P, Q respectively.

Consider the involution determined on BC by the pencil of conics having contact with p and q at P and Q. If p, q meet

BC at P_1, Q_1 then P_1, Q_1 are mates in this involution, for the pair p, q is a conic of the pencil. Also B, C are mates in this involution. The double points of this involution are therefore determined. But since PQ doubled is a conic of the pencil, the point where PQ meets BC is one of the double points of this involution.

In like manner PQ passes through one of the double points of the involution on AC determined by the pairs of mates $(A, C)(P_2, Q_2)$; P_2, Q_2 being the points where p, q meet AC.

There are thus four possible positions of PQ corresponding to

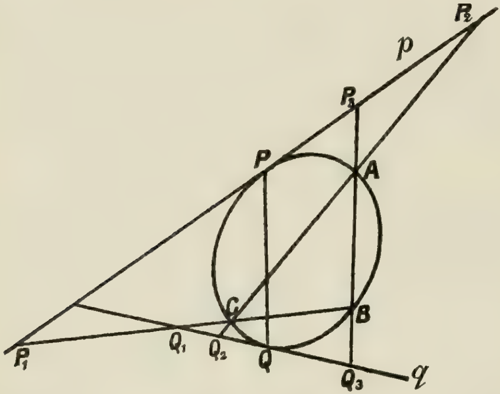


Fig. 58.

the four lines joining the double points of these two involutions, and so there are four solutions to the problem proposed.

The reader may verify that if PQ passes through double points of the involutions on BC, CA, it will also pass through a double point of the corresponding involution on AB.

Reciprocating the above construction we obtain a construction for the conics through two points and touching three lines. This, like the above, has in general four solutions.

209. Properties of confocal conics. If two of the opposite vertices of the quadrilateral *abcd* are the circular points at infinity Ω, Ω', the range of conics inscribed in this quadrilateral becomes a system of confocal conics. The involution of tangents through any point P has thus the circular lines through P for mates. Its double rays are therefore at right angles (Art. 165),

and they bisect the angles between any pair of mates (Art. 158). Such a pair of mates are the lines joining P to the two real foci S, S'. We get the series of theorems:

Through any point P of the plane two conics of a confocal system can be drawn and these cut at right angles.

The tangent and normal at any point P of a conic bisect the angles between the focal distances.

The two tangents from P to a conic are equally inclined to the rays joining P to the real foci.

Also, from the property above that double rays are at right angles, conjugate lines with regard to a system of confocal conics are perpendicular. Hence:

The locus of the poles of any straight line q with regard to a system of confocals is the normal at Q to the conic of the system touching q, Q being the point of contact of q with this conic.

210. Polar reciprocal relation between coaxial circles and confocal conics. Coaxial circles are clearly a special case of a pencil of conics, since they pass through Ω, Ω' and through two other fixed points, say A and B.

The three line-pairs of the system are

$$(AB, \Omega\Omega'), (A\Omega, B\Omega'), (A\Omega', B\Omega).$$

The first consists of the radical axis and the line at infinity; the last two are the circular lines through C and D, where C and D are the points $(A\Omega, B\Omega')$ and $(A\Omega', B\Omega)$ respectively, that is, they are by Art. 164 point-circles at C and D.

The points C, D are called the *limiting points* of the system of coaxial circles. They are imaginary if A, B are real, but real if A, B are conjugate imaginary, that is, if the radical axis does not cut the circles in real points.

Consider now the effect of taking polar reciprocals of the coaxial circles with regard to any circle of centre C. We obtain a range of conics touching the four polars of A, B, Ω, Ω' with regard to such a circle.

Now $C\Omega$ being the tangent at Ω to the circle whose centre is C, the pole of $C\Omega$ with regard to this circle is Ω. Hence the polar of A (which lies on $C\Omega$) passes through Ω. Similarly the polar of Ω' is $C\Omega'$ and the polar of B passes through Ω'. Thus A, B, Ω, Ω' reciprocate into lines ΩF, $\Omega' F$, ΩC, $\Omega' C$. The circles therefore reciprocate into conics having C, F for foci.

211. Properties of rectangular hyperbola. If two conics of a pencil are rectangular hyperbolas the points Ω, Ω' are conjugate with regard to two conics of the pencil. Therefore they are conjugate with regard to all the conics of the pencil. These are therefore all rectangular hyperbolas. Thus every conic through the intersections of two rectangular hyperbolas is a rectangular hyperbola.

If A, B, C, D be the four intersections of two rectangular hyperbolas, the line-pairs are also rectangular hyperbolas, therefore they are perpendicular. The quadrangle $ABCD$ is therefore such that pairs of opposite sides are perpendicular. Any one of its four vertices is the orthocentre of the triangle formed by the other three. It follows that any conic through the three vertices of a triangle and its orthocentre is a rectangular hyperbola (cf. Art. 118). Conversely the orthocentre of any triangle inscribed in a rectangular hyperbola lies on the curve. For if ABC be the triangle and the perpendicular through A to BC meet the hyperbola again at D, the pair AD, BC being a rectangular hyperbola, every conic through A, B, C, D is a rectangular hyperbola. But CA, BD is such a conic, therefore CA, BD are perpendicular, or D is the orthocentre of ABC.

212. Centre loci. The theorems of Arts. 202, 203 give the following results when q is the line at infinity.

The locus of the centres of a pencil of conics through four points A, B, C, D is a conic whose asymptotes are parallel to the axes of the two parabolas through the four points and which passes through the vertices of the diagonal triangle of the quadrangle $ABCD$ and the middle points of the six sides of this quadrangle.

Incidentally we have proved the theorem:

The six middle points of the sides of a complete quadrangle lie on a conic which circumscribes its diagonal triangle.

The locus of the centres of a range of conics touching four lines a, b, c, d is a straight line. Since the mid-points of the three line-pairs are evidently centres, they lie on this locus. Hence, incidentally: the middle points of the three diagonals of a quadrilateral are collinear.

213. Locus of foci of a range of conics. The involutions of tangents from two different points P, Q of the plane to the conics of a range are clearly homographically related, since either tangent through P determines uniquely the conic of

the range and therefore the pair of tangents from Q: and the converse is true if we start from Q.

These homographic involution pencils, however, have a self-corresponding ray, namely PQ, for PQ is a tangent from either P or Q to the conic of the range which touches PQ.

The locus of intersections of tangents from P and Q therefore reduces, by Art. 193, to a cubic through P and Q.

If now P and Q be taken at Ω, Ω' this locus becomes the locus of the foci of the conics of the range.

The foci of the range therefore lie on a cubic through Ω, Ω'. Such a cubic is known as a circular cubic.

If P and Q lie on any one of the three diagonals of the quadrilateral $abcd$ which defines the range, the tangents from P to the corresponding point-pair coincide along PQ; and so do the tangents from Q to the same point-pair. The homographic involutions from P and Q have therefore a pair of double rays self-corresponding. Their product breaks up into the line PQ doubled and a conic with regard to which P and Q are conjugate (Exs. XI. 13).

If P, Q be Ω, Ω', the corresponding diagonal of the quadrilateral is at infinity; the quadrilateral is a parallelogram. Hence the locus of the foci of all conics inscribed in a parallelogram is a rectangular hyperbola (for Ω, Ω' are conjugate with regard to it, by the above), together with the line at infinity doubled.

If the quadrilateral, instead of being a parallelogram, is symmetrical about a diagonal, this diagonal is obviously part of the locus. Since it does not pass through Ω, Ω' the cubic breaks up into this diagonal and a conic through Ω, Ω', that is, a circle.

Since (see Exs. VI. A, 24) the components of a point-pair are also its foci the rectangular hyperbola which is the locus of conics inscribed in a parallelogram circumscribes the parallelogram, and the circle which is the corresponding locus for the quadrilateral symmetrical about a diagonal passes through the four vertices not on this diagonal.

214. The hyperbola of Apollonius. Let the quadrangle $ABCD$, which defines a pencil of conics, be inscribable in a circle. Ω, Ω' are conjugate points in the involution determined by the pencil on the line at infinity. The double points of this involution are therefore determined by two rectangular directions. These give the axes of the two parabolas through the four points

and these are parallel to the asymptotes of the centre locus. The centre locus is then a rectangular hyperbola.

The same rectangular hyperbola is the locus of points conjugate to points at infinity with regard to the pencil of conics.

If then I^∞ be a point at infinity in the direction of one of the axes of a conic s of the pencil, the conjugate point I' is the intersection of the diameters of s and of the circle c about $ABCD$ which are conjugate to the direction of I^∞. But both of these are perpendicular to the direction of I^∞. Hence they meet at I'^∞ at infinity in the perpendicular direction. The conics of the pencil have therefore axes parallel to the asymptotes of the centre locus.

Let now I^∞ be any point at infinity, I' its conjugate point with regard to the pencil; take for s the conic of the pencil through I', and let C, O be the centres of s, c respectively.

From the property of the point I', the diameters conjugate to the direction defined by I^∞ with regard to s and c pass through I'. They are therefore CI' and OI'.

But OI' is perpendicular to its conjugate direction by a property of the circle. Thus OI' is perpendicular to the tangent at I' to the conic s, for this tangent is parallel to the diameter conjugate to CI' and therefore passes through I^∞. Hence the normal at I' to the conic through I' passes through O, or the rectangular hyperbola which is the locus of centres is also the locus of the feet of perpendiculars from O on the conics of the system.

Now let s be any conic, O any point. A circle centre O meets s in four points A, B, C, D, and considering the above results for the pencil of conics through A, B, C, D we obtain the theorem:

The feet of the normals from any point O to a conic s lie on a rectangular hyperbola through O and the centre of s, whose asymptotes are parallel to the axes of s. Since this rectangular hyperbola meets s in four points, four normals can in general be drawn from a point to a conic.

This hyperbola is known as the hyperbola of Apollonius for the point O and the conic s.

215. Joachimsthal's Theorem. Let L, M, N, K (Fig. 59) be the feet of four concurrent normals to a conic s. Consider the involution determined on the axis AA' by the pencil of conics through $LMNK$. The line-pair LM, NK determines two points P, P'. The conic s determines A, A'. The hyperbola of Apollonius, having its asymptotes parallel to the axes of s and passing

through C determines C and the point at infinity on AA'. C is then the centre of this involution. Thus

$$CP \,.\, CP' = -CA^2 \qquad\qquad (1).$$

But if T be the pole of LM with regard to s and TU be drawn perpendicular to AA', TU is the polar of P. U, P are therefore mates in the involution on AA' of conjugate points with regard to s: and A, A' are double points in this involution. Hence

$$CP \,.\, CU = CA^2 \qquad\qquad (2).$$

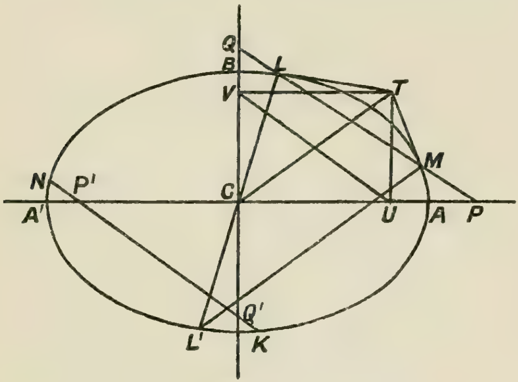


Fig. 59

From (1) and (2)

$$CP' = -CU.$$

Similarly if TV be drawn perpendicular to the other axis CB and NK meet this axis at Q'

$$CQ' = -CV.$$

Hence $P'Q'$, i.e. NK, is parallel to VU. But VU and CT, being diagonals of the rectangle $CUTV$, are equally inclined to the axes CA, CB. Therefore NK and CT are equally inclined to the axes. But CT is the diameter of s conjugate to the direction of LM. If L' be the other extremity of the diameter CL, $L'M$ (being a supplemental chord to LM) is parallel to CT and therefore equally inclined with NK to the axes. Hence by Art. 86 a circle will go through L', M, N, K. This is Joachimsthal's theorem, that if four normals to a conic be concurrent, the circle

through the feet of three of them passes also through the point diametrically opposite to the foot of the fourth.

216. Geometrical constructions for transforming any two conics into conics of given type. We have already seen (Art. 172) how to transform any two conics into circles.

Any two conics may be transformed into concentric conics by a real projection. For we have seen that there is always one side of the common self-polar triangle which is real. Projecting that

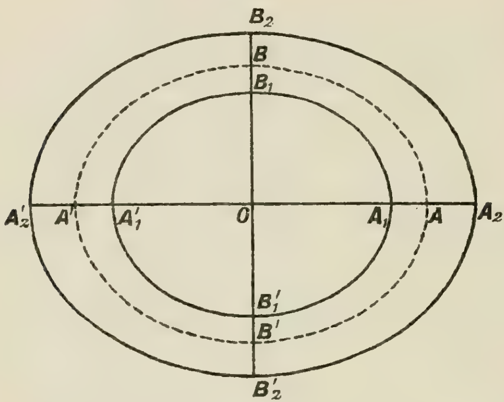


Fig. 60.

side to infinity the opposite vertex projects into the common centre.

If two vertices of the common self-polar triangle of two conics be projected into Ω, Ω', the conics project into concentric rectangular hyperbolas.

Any two conics may be projected into coaxial conics. Thus: if EFG be their common self-polar triangle, project FG to infinity and the angle FEG into a right angle.

Two coaxial conics s_1, s_2 can be transformed into one another by reciprocal polars.

Let O (Fig. 60) be their common centre, A_1A_1' and B_1B_1' the axes of s_1, A_2A_2' and B_2B_2' the axes of s_2. Find the double points A, A' of the involution determined by the pairs of mates (A_1, A_2) (A_1', A_2') and the double points B, B' of the involution

determined by the pairs of mates (B_1, B_2) (B_1', B_2'). Then from symmetry about O a conic s exists having AA', BB' as axes. Form the reciprocal polar conic s_2' of s_1 with regard to s, s_2' passes through A_2, A_2', B_2, B_2', and its tangent at A_2, being the polar of A_1 with regard to s, is perpendicular to OA_2 and so is the same as the tangent to s_2 at A_2. s_2', s_2 are thus identical.

Since any two conics may be projected into coaxial conics, projecting back we see that a conic always exists with regard to which two given conics are reciprocal polars.

217. Self-polar quadrangle. Consider two pairs of conjugate lines with regard to a conic (p, p') (q, q') (Fig. 61). Let

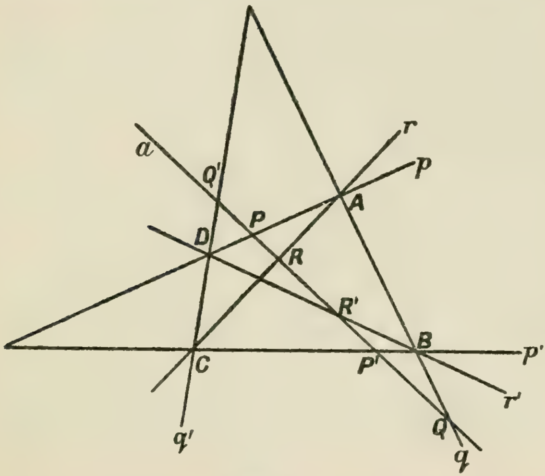


Fig. 61.

the points pq, $p'q$, $p'q'$, pq' be called A, B, C, D. Let AC, BD be r, r'. Let the polar a of A meet p, p', q, q', r, r' at P, P', Q, Q', R, R' respectively. Then because p passes through A, p is conjugate to both a and p'. Thus P' is the pole of p and P, P' are conjugate points with regard to the conic. Similarly Q, Q' are conjugate points with regard to the conic. Now by a property of the quadrangle (Art. 198) (P, P') (Q, Q') (R, R') are three pairs of mates of an involution. Hence (R, R') are mates in the involution determined by (P, P'), (Q, Q'); that is, they are conjugate points on a with regard to the conic. Also A, R' are

conjugate, since R' lies on a. Therefore R' is the pole of AR, i.e. of r: thus r, r' are conjugate lines. Hence if two pairs of opposite sides of a quadrangle are formed of conjugate lines with regard to a conic, the third pair are likewise conjugate with regard to the conic.

Such a quadrangle, of which the opposite sides are conjugate with regard to a conic, is termed *self-polar* with regard to that conic.

218. A self-polar quadrangle exists having three arbitrary points as vertices. If A, B, C be any three given points and we draw through A a line conjugate to BC and through B a line conjugate to AC these two lines intersect at the fourth vertex D of a self-polar quadrangle. D is uniquely determined unless ABC is itself a self-polar triangle, when any line through A is conjugate to BC and any line through B is conjugate to AC. In this case D may be any point of the plane. Thus a self-polar triangle and any fourth point form a self-polar quadrangle.

If two vertices of a self-polar quadrangle be conjugate, they form with a third vertex of the quadrangle a self-polar triangle. For let A and B be two conjugate points, a and b two opposite sides of the self-polar quadrangle through A, B respectively. Then a, b are conjugate. Hence either A is the pole of b or the pole of b is a point P on a other than A. Now the polar of A passes through B and the polar of P is b and also passes through B. B is then the pole of AP, i.e. of a. Therefore either A is the pole of b or B is the pole of a. Now suppose A is the pole of b and C is the vertex of the quadrangle on b. Then AB, CD being conjugate the pole of AB is on CD. But it lies also on b, since this is the polar of A. Therefore C is the pole of AB and ABC is a self-polar triangle.

219. Two self-polar quadrangles having two vertices common are inscribed in the same conic. Let $ABCD$, $ABC'D'$ be two self-polar quadrangles. Then AC, AD, AC', AD' are conjugate to BD, BC, BD', BC' respectively. And conjugate pencils being projective

$$A(CDC'D') \barwedge B(DCD'C')$$
$$\barwedge B(CDC'D') \text{ by Art. 21.}$$

Therefore C, D, C', D' are intersections of corresponding rays of two projective pencils of four rays through A and B, that is, they lie on a conic through A and B, which proves the theorem required.

220. Conics harmonically circumscribed to a conic. Let s be any conic, $ABCD$ any quadrangle self-polar for s, s' a conic circumscribed about the quadrangle $ABCD$, $A'B'C'D'$ another quadrangle self-polar with regard to s, of which three vertices A', B', C' are taken on s'. Consider a self-polar quadrangle three of whose vertices are A', B, C; by the previous theorem its fourth vertex D_1 also lies on s'; next consider the quadrangle three of whose vertices are A', B', C; its fourth vertex D_2 lies on s'; and finally the fourth vertex D' of the quadrangle three of whose vertices are A', B', C', lies on s'.

Thus if one quadrangle exist self-polar with regard to one conic s and inscribed in another s' an infinity of such quadrangles exist, any three of whose vertices may be given on s' arbitrarily.

By taking any vertex A' arbitrarily and for a second vertex one of the points at which the polar of A' with regard to s meets s' we have two vertices of the self-polar quadrangle conjugate, and so obtain a self-polar triangle of s inscribed in s'. Again if any self-polar triangle of s be given and s' be any conic circumscribing it, the addition of a point on s' transforms the self-polar triangle into a self-polar quadrangle with regard to s inscribed in s' and the previous theorems hold of two such conics s and s'.

The results obtained may be summarised as follows: if two conics s and s' be such that either one self-polar quadrangle or one self-polar triangle of s is inscribed in s', an infinity of both triangles and quadrangles, self-polar with regard to s, can be inscribed in s', one vertex of each of the triangles and three vertices of each of the quadrilaterals, being arbitrary.

The conic s' is then said to be *harmonically circumscribed* to the conic s*.

221. Triangles self-polar with regard to the same conic. It follows from the above that if $A_1B_1C_1$, $A_2B_2C_2$ are two triangles self-polar with regard to a conic s, their six vertices lie on a conic s'.

For take s' as the conic through A_1, B_1, C_1, A_2, B_2. The quadrangle $A_1B_1C_1A_2$ is self-polar with regard to s. Complete the self-polar quadrangle three of whose vertices are C_1, A_2, B_2. Since $C_1A_2B_2$ is *not* a self-polar triangle, and A_2, B_2 are conjugate points, this fourth vertex forms with A_2, B_2 a self-polar triangle (Art. 218). It is therefore C_2, and it lies on s' by Art. 220.

* In Reye's notation s' *supports* or *carries* s.

In particular if s be a rectangular hyperbola whose centre is O, the triangle $O\Omega\Omega'$ is self-polar with regard to s. If ABC be any other self-polar triangle O, Ω, Ω', A, B, C lie on a conic, that is, O lies on the circle circumscribing ABC, or: the locus of the centres of rectangular hyperbolas for which a given triangle is self-polar is the circumscribing circle of the triangle.

222. Conics harmonically inscribed in a conic. Reciprocating the theorems of Arts. 217—221 we can define a self-polar quadrilateral with regard to a conic s as one whose pairs of opposite vertices are conjugate with regard to s.

Also if one self-polar quadrilateral or triangle of s be circumscribed about another conic s', an infinity of both triangles and quadrilaterals, self-polar with regard to s, can be circumscribed about s': one side of each triangle, or three sides of each quadrilateral, may be arbitrarily given.

s' is then said to be *harmonically inscribed* in s.

Since any two conics may be transformed into one another by reciprocal polars (Art. 216), applying such a transformation to transform the conic s' of the present Article into s, the self-polar quadrilaterals and triangles with regard to s circumscribed about s' reciprocate into self-polar quadrangles and triangles with regard to s', inscribed in s. Hence if s' be harmonically inscribed in s, then s is harmonically circumscribed to s' and conversely.

We find as in the last Article that the six sides of two triangles self-polar with regard to the same conic touch a conic.

223. Triangles inscribed in a conic are self-polar with regard to a conic. Conversely we have to show that if two triangles $A_1B_1C_1$, $A_2B_2C_2$ are inscribed in a conic s', there exists a conic s with regard to which they are both self-polar.

We will first show that a conic s exists with regard to which a given triangle $A_1B_1C_1$ is self-polar and a given point and line, A_2 and B_2C_2, are pole and polar. For let A_1A_2, B_2C_2 meet B_1C_1 at L, M (Fig. 62), then M is the pole of A_1A_2 and L, M are conjugate points with regard to the conic required. The double points P, Q of the involution of which (B_1, C_1) (L, M) are pairs of mates give the intersections of this conic with B_1C_1. Join A_2P meeting B_2C_2 at T and let R be harmonically conjugate to P with regard to A_2T. Describe the conic touching A_1P, A_1Q at P and Q respectively and passing through R.

This is the conic s required. For A_1 is the pole of B_1C_1 with regard to s and B_1, C_1 being harmonically conjugate with regard to P, Q $\therefore$ $A_1B_1C_1$ is a self-polar triangle for s. Also (L, M) and (A_1, M) being clearly pairs of conjugate points with regard to s (since M is on the polar of A_1 and $\{PLQM\} = -1$), M is the pole of A_1L and M, A_2 are conjugate. Also, because $\{A_2PTR\} = -1$, A_2, T are conjugate. Hence TM, that is B_2C_2, is the polar of A_2 with regard to s.

It now follows at once that $A_2B_2C_2$ is self-polar with regard to s. For if this were not so, let C_2' be the point conjugate to B_2

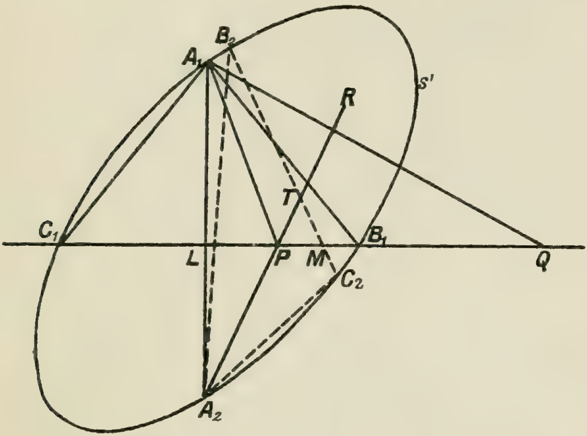


Fig. 62.

on B_2C_2. Then $A_1B_1C_1$, $A_2B_2C_2'$ are self-polar for s. That is, C_2' lies on the conic through A_1, B_1, C_1, A_2, B_2, i.e. on s'. Hence C_2' is at C_2.

In like manner (or by reciprocation) it may be shown that triangles circumscribed to the same conic are self-polar for a conic.

Combining these results with those of Arts. 221, 222 we obtain the theorem: if two triangles are inscribed in a conic, they are circumscribed to a conic (cf. Exs. V. A, 5). This may be put into the form: if one triangle exist which is inscribed in one conic s and circumscribed to another s', an infinite number of such triangles exist.

For let $A_1B_1C_1$ be such a triangle; through any point A_2 of s we can draw two tangents to s' meeting s in B_2C_2. Then the triangles $A_1B_1C_1$, $A_2B_2C_2$ are inscribed in s: their six sides therefore touch a conic. But five of these sides touch s'. Hence the sixth side touches s'.

224. Conics harmonically inscribed and circumscribed to the same conic. Let s_1, s_2 be two conics such that a triangle ABC can be inscribed in s_1 and circumscribed to s_2. Let PQR be the common self-polar triangle of s_1, s_2. Let $A'B'C'$ be a second triangle inscribed in s_1 and circumscribed to s_2. Such a triangle exists by the last Article. Also a conic s exists for which ABC, $A'B'C'$ are self-polar.

Apply a transformation by reciprocal polars with s as base conic. The triangles ABC, $A'B'C'$ being self-polar with regard to s, transform into themselves. The conic s_1 which is circumscribed to them transforms into the conic s_2 which is inscribed in them. The triangle PQR which is self-polar with regard to s_1 and s_2 transforms into a triangle self-polar with regard to s_2 and s_1, that is, it transforms into itself. It is therefore also self-polar with regard to s. Hence PQR, ABC are two self-polar triangles with regard to s. Thus P, Q, R, A, B, C are six points on a conic.

It follows from the above demonstration that two conics s_1 and s_2, such that a triangle exists inscribed in s_1 and circumscribed to s_2, are also such that a conic s exists harmonically inscribed in s_1 and circumscribed to s_2.

If in the above theorem the conics s_1, s_2 are coaxial, their common self-polar triangle is formed by the common axes and the line at infinity. If then ABC is a triangle inscribed in s_1 and circumscribed to s_2 a conic can be drawn through A, B, C passing through the centre of s_1 and having its asymptotes parallel to the axes. But this is a hyperbola of Apollonius for s_1. Hence if such a triangle exist the three normals to s_1 at its vertices A, B, C are concurrent.

225. Pencil of conics harmonically circumscribed to a conic. If two conics s_1, s_2 of a pencil are harmonically circumscribed to the same conic s, let three of their common points A, B, C be taken for the vertices of a quadrangle self-polar for s. Then by the property of the harmonically circumscribed conics, the fourth vertex D of this quadrangle lies on both s_1 and s_2. The four points of intersection of such conics then form

a quadrangle self-polar with regard to s and so every conic of the pencil through them is harmonically circumscribed to s.

In like manner if two conics of a range are harmonically inscribed in the same conic s, all the conics of the range are harmonically inscribed in s.

226. **Faure and Gaskin's Theorem.** If we take two circles harmonically circumscribed to a conic s their intersections form a quadrangle self-polar for s. Hence their radical axis is conjugate with regard to s to the opposite side of this quadrangle, namely the line at infinity. This radical axis therefore passes through the centre of s. The tangents from the centre of s to all circles harmonically circumscribed to s are then equal, that is, a circle concentric with s cuts orthogonally every circle harmonically circumscribed to s. This circle is therefore the locus of the point circles harmonically circumscribed to s. But a point circle is simply two circular lines, and a line-pair harmonically circumscribed to s reduces to a pair of conjugate lines for s. Hence the above locus is the locus of points the circular lines through which are conjugate for s, that is, the orthoptic circle of s by Art. 168.

Hence every circle circumscribing a triangle self-polar with regard to a conic cuts orthogonally the orthoptic circle of the conic.

227. **Nets and webs of conics.** The set of conics harmonically circumscribed to one, two, three or four conics are said by Reye to form a *net* of the fourth, third, second or first grade respectively, and the set of conics harmonically inscribed in one, two, three or four conics are said to form a *web* of the fourth, third, second or first grade respectively.

The net and web of the first grade are identical with the pencil and range of conics respectively. For consider the condition that a conic shall be harmonically circumscribed to a given conic. Through any four points A, B, C, D of the plane such a conic can be drawn, for it is the conic through A, B, C, D, E where E is the fourth vertex of the quadrangle of which A, B, C are three vertices and which is self-polar with regard to the given conic. The condition that a conic shall be harmonically circumscribed to a given conic leaves it free to pass through four given points in one way only. It must then be equivalent to one linear relation between the coefficients of the conic.

Hence a conic belonging to a net of the first grade has already to satisfy four linear relations between its coefficients. One such conic can then be made to pass through any point of the plane.

But the intersections of two such conics form a quadrangle self-polar with regard to each of the four conics determining the net. The conic through any point P and the vertices of this quadrangle is a conic of the net; and it is the only conic of the net through P. The net therefore reduces to a pencil of conics. Similarly the web of the first grade reduces to a range.

It follows from the above considerations that a net of the second, third, or fourth grade can be made to pass through two, three, or four given points in the plane, and that a web of the second, third, or fourth grade can be made to touch two, three, or four given lines respectively.

In general when the terms net and web are used without further qualification, a net or web of the second grade is meant.

228. Similar conics. Two coplanar conics are said to be similar and similarly situated if they can be brought into a plane perspective relation in which the axis of collineation is at infinity. It follows that the points at infinity of two such conics coincide, or their asymptotes are parallel.

Conversely if two conics s_1, s_2 have their asymptotes parallel they are similar and similarly situated. For let I^∞, J^∞ be their two common points at infinity, t one of their common tangents touching s_1 at P_1, s_2 at P_2. Through P_1, P_2 draw any two parallel lines meeting s_1, s_2 at Q_1, Q_2 respectively, and let Q_1, Q_2 meet t at O. With O as pole of perspective, the line at infinity as axis of collineation and P_1, P_2 as a pair of corresponding points, construct the conic s_2' in plane perspective with s_1. Then s_2, s_2' have in common the points I^∞, J^∞, Q_2, P_2 and the tangent at P_2. They are therefore identical, that is, s_1, s_2 are similar and similarly situated.

Any two conics may be projected into similar conics by projecting one of their common chords to infinity. Projecting back we see that any two conics may be brought into plane perspective by taking any one of their common chords as axis of collineation.

EXAMPLES XII.

1. Show that the cross-ratio of the flat pencil formed by the polars of a point U with regard to four conics of a pencil is independent of the position of U in the plane.

2. Show that the cross-ratio of the range formed by the poles of a line u with regard to four conics of a range is independent of the position of u in the plane.

3. Prove that the harmonic conjugates of any point with regard to four given pairs of points in involution have a constant cross-ratio.

4. Show that the circles on the three diagonals of a quadrilateral as diameters have a common radical axis.

[For these circles are orthoptic circles of the point-pairs of the range.]

5. Show that in a range of conics two are rectangular hyperbolas and that their orthoptic circles are the point-circles of the system of coaxial orthoptic circles of the range.

6. Prove that if two conics of a pencil have their axes parallel, all the conics of the pencil have their axes parallel and one of these conics is a circle.

7. Show that in the construction of Art. 210 it is possible to choose the radius of the circle with regard to which we reciprocate so that both limiting points shall reciprocate into foci.

8. Show how to construct the centre of a conic touching five given lines.

9. The locus of centres of rectangular hyperbolas circumscribing a triangle is the nine-points circle of the triangle.

10. If a pencil of conics circumscribe a rectangle, show that the eleven-point conics are rectangular hyperbolas.

11. The centre of the locus of centres of conics of a pencil is the centroid of the quadrangle defining the pencil.

[For the centre-locus passes through the mid-points P, Q, R, S of AB, BC, CD, DA. But $PQRS$ is readily shown to be a parallelogram. Hence the intersection of PR, QS, which is the centroid of the quadrangle, is also the centre of the centre-locus.]

12. A system of conics have a given circle of curvature at A and pass through B. Show that the locus of their centres is a conic whose centre divides AB in the ratio 1 : 3.

13. If a point U describes a normal to a conic the feet P, Q, R of the three other normals drawn from U to the conic form a triangle circumscribing a parabola touching the axes.

[If T (Fig. 59) describe LT, the ranges $[U]$, $[V]$, and $\therefore$ $[P']$, $[Q']$ are similar $\therefore$ NK envelops a parabola.]

14. $A_1B_1C_1$, $A_2B_2C_2$ are two triangles inscribed in the same conic. Conics s_1, s_2 are described about $A_1B_1C_1$, $A_2B_2C_2$ having double contact with one another. Show that their common chord of contact touches the conic for which $A_1B_1C_1$, $A_2B_2C_2$ are self-polar.

15. If a conic s_1 be harmonically inscribed in a conic s_2, two points of s_2, the tangents at which are conjugate with regard to s_1, lie on a tangent to s_1. Conversely if P be the pole with regard to s_2 of a tangent to s_1 the tangents from P to s_2 are conjugate with regard to s_1.

16. If a conic s_1 be harmonically circumscribed to a conic s_2, two tangents to s_2 whose points of contact are conjugate with regard to s_1, intersect on s_1. Conversely the polar with regard to s_2 of a point of s_1 meets s_2 in points conjugate with regard to s_1.

17. From the theorem that if two triangles are circumscribed to a conic they are inscribed in another conic prove, by taking two vertices of one triangle to be the circular points at infinity, that the circle circumscribing a triangle formed by three tangents to a parabola passes through the focus.

18. The locus of the centre of a rectangular hyperbola which is inscribed in a given triangle is the circle for which the triangle is self-polar.

[For the hyperbola is harmonically inscribed in the circle, i.e. triangles self-polar for the hyperbola are inscribed in the circle, ∴ by Art. 221 the centre of hyperbola lies on this circle.]

19. If a circle cut harmonically the sides of a triangle circumscribed to a conic, it cuts orthogonally the orthoptic circle of the conic.

[For the circle is harmonically circumscribed to the conic.]

20. If triangles exist which are inscribed in a circle c and circumscribed to a circle c' it is necessary and sufficient that the rectangle contained by segments of chords of c through centre of c' should be numerically equal to twice the product of the radii.

21. Show how, by a real projection, to project two conics which intersect in only two real points into two similar and similarly situated ellipses.

22. From a given point A a variable chord APQ is drawn to a given conic s. Through P, Q and another given point B a conic is drawn similar and similarly situated to s. Prove that this conic passes through a certain fixed point other than B.

[Project the points at infinity on s into the circular points.]

23. If two coaxial conics be such that a triangle exist which is circumscribed to one conic and inscribed in another, prove that the axes and the sides of the triangle touch a parabola.

CHAPTER XIII.

THE CONE AND SPHERE.

229. The Geometry of the sheaf. A *sheaf* is a figure formed by lines and planes through a point O which is called the centre or vertex of the sheaf. The lines and planes of the sheaf cut an arbitrary plane in the points and lines respectively of a plane figure. To the points of any curve in this plane correspond the generators of a cone of vertex O standing upon this curve as base. To the tangents to this curve correspond tangent planes to the cone. The correspondence between the lines of the sheaf and the points of the plane figure is thus a one-one correspondence which preserves properties of incidence and tangency. It also preserves anharmonic properties, for to a range of four points in the plane corresponds in the sheaf a flat pencil having the same cross-ratio; and to a flat pencil of four rays in the plane corresponds in the sheaf an axial pencil having the same cross-ratio, and whose axis passes through the vertex of the flat pencil in the plane.

It follows that all projective properties of plane figures, that is, properties of incidence, tangency and cross-ratio, lead to corresponding properties of the sheaf. In what follows I shall enumerate the most important of these properties. The proofs will in general be obvious from the above principles; where they are not so obvious, a few hints will be given to enable the student to supply the demonstration for himself.

230. Projective properties of the cone of second order. A cone of second order is defined as one which is cut in two points only by any straight line u. Joining u to the vertex O of the cone we see that the cone is cut by any plane through its vertex in two lines of the sheaf through O. Hence its intersection with any plane is a curve of the second degree or conic.

Conversely to a conic in the plane corresponds a cone of the second order in the sheaf.

Chasles' Theorem gives: if α be a fixed tangent plane to a cone of vertex O, x a fixed generator, π a variable tangent plane, p its generator of contact, the flat pencil $\alpha(\pi)$ is homographic with the axial pencil $x(p)$.

We deduce, as in Chap. III, that a variable tangent plane π cuts four fixed tangent planes in a flat pencil of constant cross-ratio; or taking any tangent line t lying in π, t cuts these four tangent planes in a range of constant cross-ratio.

Also a variable generator to the cone determines with four fixed generators an axial pencil of constant cross-ratio.

Conversely the product of two homographic axial pencils whose axes x_1, x_2 intersect at O is a cone of the second order vertex O, having x_1x_2 for a pair of generators, and the envelope of the planes determined by the corresponding rays of two homographic flat pencils having a common vertex but not lying in the same plane is again such a cone.

If two homographic axial pencils have a self-corresponding plane, their axes intersect in this plane. The flat pencils in which they intersect any plane are perspective and the product of the two homographic axial pencils is a plane, together with the self-corresponding plane.

Pascal's Theorem gives: if a six-faced solid angle be inscribed in a cone of second order, the lines of intersection of opposite faces are coplanar.

Brianchon's Theorem gives: if a six-faced solid angle be circumscribed to a cone of the second order, the planes joining opposite edges pass through a line. By taking arbitrary points on the six edges of this solid angle and joining them we obtain the somewhat different enunciation: if a skew hexagon be circumscribed to a cone of the second order, the three diagonals joining opposite vertices intersect a line through the vertex of the cone.

231. Pole and polar properties of the cone of second order. To any line p through the vertex of a cone of second order corresponds a plane π through the vertex which is called the diametral plane of the cone conjugate to the diameter p. These are obtained by joining to the vertex of the cone the polar of the point in which p cuts any plane α with regard to the section of the cone by α. Also since cross-ratio is unaltered by projection, if P be any point of p and a line through

P meet the cone at Q, R and π at P' then P, P' are harmonically conjugate with regard to Q, R. π is therefore also called the polar plane of P, and conversely P is a pole of π. We see that any plane through the vertex has an infinite number of poles, which all lie on its conjugate diameter.

If P' lies on the polar plane of P, conversely the polar plane of P' passes through P.

Such planes are called conjugate diametral planes. They meet any plane in two lines which are conjugate with regard to the section of the cone by that plane. From the property that two such conjugate lines are harmonically conjugate with regard to the two tangents from their intersection, we see that two conjugate diametral planes for the cone are harmonically conjugate with regard to the two tangent planes through their intersection.

Similarly conjugate diameters of the cone are harmonically conjugate with regard to the two generators of the cone in their plane.

The polar plane of the vertex is indeterminate: conversely every plane not passing through the vertex has the vertex for pole.

To a triangle self-polar with regard to any plane section of a cone of second order corresponds a trihedral angle or *three-edge* self-polar with regard to the cone, the edges of which pass through the vertices of the triangle. These edges form a set of three diameters conjugate pair and pair, and such that each is conjugate to the opposite face of the three-edge.

Chords parallel to any diameter of the cone (i.e. any line through the vertex) are bisected by the conjugate diametral plane. This plane also contains the generators of contact of the two tangent planes to the cone through the diameter in question.

A diameter of the cone passes through the centres of the sections of the cone by planes parallel to its conjugate diametral plane.

For let u, v, w be a self-polar three-edge, it will meet a plane parallel to vw in the vertices U, V^∞, W^∞ of a self-polar triangle for the section. U is thus the pole with regard to this section of $V^\infty W^\infty$, that is, of the line at infinity in the plane of the section. U is therefore the centre of the section.

232. Cones harmonically inscribed in and circumscribed to other cones. If a cone C_1 be circumscribed about a three-edge self-polar for another cone C, it contains an infinity

of such three-edges and is said to be harmonically circumscribed to C. If a cone C_2 be inscribed in a three-edge self-polar for C it is inscribed in an infinity of such three-edges and is said to be harmonically inscribed in C.

If C_1 be harmonically circumscribed to C, then C is harmonically inscribed in C_1.

233. The complete four-edge and four-face. Corresponding to a complete quadrangle in the plane we have a complete *four-edge* in the sheaf. Such a four-edge has six faces and the meets of the pairs of opposite faces form its diagonal three-edge.

Similarly to the complete quadrilateral corresponds the complete *four-face*, with six edges and three diagonal planes, which form its diagonal three-edge. The harmonic properties of the complete quadrangle and quadrilateral are transferred at once to the four-edge and four-face. Thus two faces of the diagonal three-edge of a complete four-edge are harmonically conjugate with regard to the two faces of the four-edge through their intersections; and two edges of the diagonal three-edge of a complete four-face are harmonically conjugate with regard to the two edges of the four-face in their common plane.

234. The circle at infinity. Any sphere S cuts the plane at infinity τ^∞ in an imaginary circle. Consider any plane π. This cuts S in a circle. The points where this circle meets the line at infinity of π are the circular points at infinity of π. They are on the sphere S and therefore on its circle of intersection with the plane at infinity. This circle of intersection is therefore the locus of the circular points at infinity in all planes. It is called the *circle at infinity*, and all spheres contain it.

235. The spherical cone. The cone formed by joining any point O to the points of the circle at infinity will be called the *spherical cone* through O. Every plane π meets a spherical cone in a circle. For take the line at infinity of π. It meets the cone on the circle at infinity. The two circular points at infinity of π are therefore on the section: hence the latter must be a circle.

Such a cone, being a surface of the second order passing through the circle at infinity, is to be also considered as a sphere. It is, in fact, a *point-sphere* and is the limiting case of a sphere of vanishingly small radius—precisely as a pair of circular lines form a point-circle. Hence the name spherical cone.

236. Every self-polar three-edge of a spherical cone is rectangular. Let u, v be two conjugate diameters of the cone. Let the plane uv meet the cone in generators s, t. Since the cone is spherical the points at infinity of s, t are the circular points at infinity of their plane. But u, v are harmonically conjugate with regard to s, t. Hence (Art. 165) u, v are at right angles.

Thus any two conjugate diameters of a spherical cone are perpendicular and a self-polar three-edge for such a cone is rectangular.

Since two conjugate diametral planes can always be taken to form two faces of a self-polar three-edge, it follows that conjugate diametral planes of the spherical cone are perpendicular.

237. Principal axes and diametral planes of a cone of the second order. Since two conics have one common self-polar triangle, by considering the two cones having the same vertex and standing on these conics as bases, we see that two cones with a common vertex always have one common self-polar three-edge. Taking one of the cones to be spherical we see that any cone of the second order has one rectangular self-polar three-edge. The faces of this three-edge are then clearly by Art. 231 planes of symmetry for the cone, and any one of them cuts a section made by a plane parallel to another in an axis of this section.

If, however, two cones have double contact, the conics in which they intersect any plane have also double contact, and have an infinity of common self-polar triangles with a common vertex. The cones have therefore an infinity of common self-polar three-edges with a common edge.

We deduce that if a cone has double contact with the spherical cone having the same vertex, it has an infinity of rectangular self-polar three-edges with a common edge and therefore (because trirectangular) with their other edges all coplanar. A plane perpendicular to this common edge will meet the given cone in a conic for which all pairs of conjugate diameters are perpendicular, that is, in a circle, through the centre of which the common edge in question passes. The cone is then a right circular cone, the common edge of the self-polar trirectangular three-edges being the axis of the cone.

A right circular cone has therefore double contact with the spherical cone, and conversely if a cone have double contact with the spherical cone, it is right circular.

238. Focal lines of a cone. Consider the intersection of any given cone and the spherical cone of same vertex by any plane π. The two conics s, t forming the intersection have four common tangents, which meet in six points. Hence the two cones have four common tangent planes which meet in six lines. These six lines are called *focal lines* of the cone.

Any two planes through a focal line which are harmonically conjugate with regard to the two common tangent planes through that line are conjugate with regard to both cones. Hence they are perpendicular by Art. 236. The focal lines have therefore the property that conjugate diametral planes through them are perpendicular. The focal lines are situated in pairs on the three principal diametral planes. For they form the six edges of a complete four-face whose diagonal three-edge is the common self-polar three-edge of the given cone and the spherical cone, that is, the principal three-edge of the given cone.

Any pair of conjugate diametral planes of the given cone which are perpendicular are also conjugate for the spherical cone (Art. 236). They meet any plane π in two lines l, m which are conjugate for both conics s, t and therefore for all the conics of the range determined by s, t. They divide therefore harmonically the three point-pairs of this range. But these point-pairs are the intersections with π of the three pairs of focal lines. A pair of focal lines are therefore harmonically separated by a pair of perpendicular conjugate diametral planes.

Taking for the perpendicular conjugate diametral planes the tangent and normal planes through a generator, we see that these latter planes bisect the dihedral angles between the planes through this generator and any pair of focal lines. The student should compare this property with that of the tangent and normal to a conic which bisect the angles between the focal distances.

Again, the planes which bisect the dihedral angles between two tangent planes α, β to the cone through a diameter are clearly conjugate perpendicular planes. Thus they also bisect the dihedral angles between the two planes σ, τ through this diameter and a pair of focal axes.

The dihedral angle between α and σ is thus equal to the dihedral angle between β and τ.

239. Cyclic planes of a cone. The conics s, t of the last Article, in which a given cone and the spherical cone of same vertex meet a plane π, have four common points, determining three line-pairs.

Any two points conjugate with regard to both the conics s, t are likewise conjugate with regard to these line-pairs.

It follows that the given cone and the spherical cone have four common generators, determining three pairs of diametral planes which separate harmonically any two diameters conjugate for both cones, that is, any two perpendicular conjugate diameters of the given cone.

These six planes are called the *cyclic planes* of the given cone. Any plane parallel to them contains two of the common points at infinity of the cone and the spherical cone, that is, the section of the cone by this plane passes through the circular points at infinity of the plane and is a circle. Thus the cyclic planes are planes parallel to the planes of circular section.

A pair of cyclic planes pass through an edge of the common self-polar three-edge of the two cones, that is, through an axis; and since they separate harmonically the two other axes (these being perpendicular conjugate diameters), their traces on the plane of those axes are equally inclined to those axes. Hence a pair of cyclic planes pass through an axis and are equally inclined to the principal diametral planes through that axis.

Consider any plane α through the vertex O of the cone, meeting the cone in two generators a, b and a pair of cyclic planes in rays x, y. The conjugate diameters in α form an involution flat pencil. Let u, v be rectangular rays of this pencil. Then x, y and a, b are harmonically conjugate with regard to u, v, therefore x, y are equally inclined to u, v. Similarly a, b are equally inclined to u, v. Hence the angle between a and x is equal to the angle between b and y.

240. Real focal lines and cyclic planes. Since the two circular points at infinity in any plane are conjugate imaginary it follows that the conjugate imaginary point to any point on the circle at infinity itself lies on the circle at infinity. The circle at infinity is therefore its own conjugate imaginary locus*, and the same holds clearly for a spherical cone with a real vertex.

* The student may ask why this does not make it a real circle, since it was stated on the same grounds, in Art. 125, that a straight line which was its own conjugate imaginary was real. The answer is that the circle at infinity is indeed determined by two *real* equations, namely that of any sphere and of the line at infinity. But the locus determined by such real equations need not itself be real, unless the equations are both linear, which is the case for the straight line.

The reasoning used in Art. 125 therefore applies here, to show that the common generators and the common tangent planes of a real cone and a spherical cone having the same vertex are conjugate imaginary in pairs. This shows, bearing in mind the corresponding results of Art. 205 that: (a) the common self-polar three-edge is entirely real, or the three principal planes and axes of any cone are real; (b) of the three pairs of focal lines, one only is real, the other two pairs being imaginary; (c) of the three pairs of cyclic planes, one only is real.

241. Representation of the sheaf on a sphere. If we describe a sphere of arbitrary radius, whose centre is the vertex of a sheaf, every plane of the sheaf determines a great circle on the sphere and every line of the sheaf a point on the sphere. An axial pencil of planes determines a spherical pencil formed by great circles passing through a point, and of course passing also through its antipodal point.

A flat pencil of lines of the sheaf determines a spherical range of points on a great circle. The range is really a twin range, since each line of the pencil meets the sphere in two antipodal points. The cross-ratio of four elements is determined from four arcs of a spherical range, or from four angles of a spherical pencil, by a formula involving the sines of these arcs or angles, identical with that proved for flat pencils in Art. 22. Also all great circles meet a spherical pencil of four great circles in spherical ranges of the same cross-ratio. We have thus a whole theory of projective and perspective forms of the first order on the sphere which corresponds to the theory already developed for the plane. There are certain differences, for example, bearing in mind that two points of a spherical range correspond to one line of the defining flat pencil through the centre of the sphere, we see that there are *two* points of a spherical range determining a given cross-ratio with three given points of the range.

Also the principle of duality will hold for spherical figures. For since the angles between two great circles are equal to the arcs joining their poles (measured by the angles subtended at the centre), if we make a great circle correspond to its pair of poles and conversely, we have spherical pencils corresponding to equi-anharmonic spherical ranges and conversely.

242. Sphero-conic. The twin curve in which a cone of the second order meets a concentric sphere is called a sphero-conic. The properties of sphero-conics are merely a restatement in suitable language of the properties of the cone of second order.

The student will find it a useful exercise to trace and tabulate them.

The following two may be stated:

There are two real pairs of antipodal foci S, S', H, H' of a sphero-conic, such that every pair of great circles conjugate with regard to the sphero-conic through any one of these foci are rectangular.

If P (Fig. 63) be any point on a sphero-conic, the focal distances SP, HP equally inclined to the tangent and normal great circles at P.

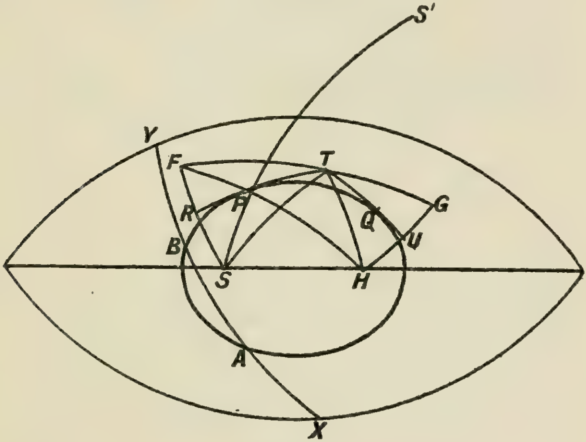


Fig. 63.

If TP, TQ be tangent great circles to a sphero-conic from T, the angle STP is equal to the angle HTQ. The above properties follow from Art. 238.

There are two real cyclic lines for a sphero-conic such that if any great circle meet the sphero-conic at A, B and the cyclic lines at X, Y, then arc AX = arc BY (Art. 239). These cyclic lines play therefore, to some extent, the part of asymptotes.

To prove that if P be any point of a sphero-conic, S, H two non-antipodal foci, then the sum or difference of the arcs SP, HP is constant.

Let TP, TQ, as before, be two tangent great circles to the sphero-conic meeting at T. Let F, G be the points symmetrical with S, H with regard to TP, TQ respectively.

The angles FTP, PTS, HTQ, QTG are all equal.

$\therefore$ angle $FTS =$ angle HTG.

Adding angle STH we have angle $FTH =$ angle STG, arc $FT =$ arc ST and arc $TH =$ arc TG.

The spherical triangles FTH, STG are congruent and arc $FH =$ arc SG.

Now if the tangent great circle PT move round the curve, TQ remaining fixed, G remains fixed and SG remains fixed, $\therefore$ arc $FH =$ a constant length.

Joining PF, PS, PH, angle $FPR = SPR$ by symmetry; and $SPR = TPH$ (equal inclination of focal distances to tangent). Hence angle $FPR = TPH$ or FPH is a great circle. Thus $SP + PH = FP + PH = FH =$ constant.

If we take two foci S', H not inside the same oval, then $SP =$ semi-circumference $- S'P$. Thus $PH - S'P =$ constant.

EXAMPLES XIII.

1. Show that properties of a spherical figure may be reciprocated by making a great circle correspond to its poles and conversely. Either figure may then be called the polar figure of the other.

Prove that the polar of a sphero-conic is a sphero-conic, the cyclic lines of the one reciprocating into the foci of the other.

Show that this is a particular case of ordinary polar reciprocation with regard to a spherical cone.

2. Show that if a pair of tangent planes through a diameter d of a cone of second order touch the cone along s, t and f be a focal line, the planes fs, ft are equally inclined to fd.

Deduce that tangents to a sphero-conic subtend equal or supplementary angles at a focus.

3. Find the envelope of a plane of a sheaf which moves so that its traces on two fixed planes of the sheaf subtend a fixed dihedral angle at a fixed line of the sheaf.

4. If a tangent plane to a cone of the second order meet the tangent planes perpendicular to a principal diametral plane in lines x, y, the lines x, y subtend a right dihedral angle at a focal line situated in the given principal diametral plane.

State the corresponding theorem for the sphero-conic.

5. If a principal diametral plane of a cone of the second order meet the cone in a, a' and p be any generator of the cone, the planes pa, pa' meet either cyclic plane through the axis perpendicular to the given principal plane in two lines at right angles.

6. Show that a plane perpendicular to a focal axis cuts the cone in a conic, the focal axis in question passing through a focus of this conic.

7. Through any ray of a sheaf through O two cones of a confocal system having O for vertex can be described and these are orthogonal.

8. Show that if one rectangular three-edge can be inscribed in or circumscribed to a cone of the second order, an infinite number of such three-edges can be so inscribed or circumscribed.

9. If one side of a spherical triangle move so that the area of the triangle remains constant, it envelops a sphero-conic of which the other two sides (which remain fixed) are the cyclic lines.

[Deduce from Art. 242 by the method of Ex. 1.]

10. If two non-coplanar conics touch one another a cone of the second order passes through both of them.

[Use Arts. 12, 43.]

11. Two cones of the second order which touch one another along the line joining their two vertices intersect in a conic.

12. Prove that any sphere which passes through a section of a cone made by a plane parallel to one cyclic plane meets the cone again in a plane section parallel to the other cyclic plane of the pair.

CHAPTER XIV.

QUADRICS.

243. Order, Class, Degree. The *order* of a surface is the number of points in which it is met by any straight line not lying in it.

The *class* of a surface is the number of tangent planes which may be drawn to it through any straight line not lying in it.

The *degree* of a skew or twisted curve (that is, a curve which does not lie in a plane) is the number of points in which it is met by any plane.

Note that this definition of degree coincides with the one previously adopted for a plane curve, for the intersections of the latter with any straight line in its plane may also be looked upon as intersections with another plane through this straight line.

Note also that a plane section of a surface of the nth order is a curve of the nth degree.

The following result will be assumed: three surfaces of order m, n, p intersect in mnp points, real or imaginary. This is evident from analytical considerations if we remember that the equation to a surface of order n is of the nth degree in the coordinates.

244. Developables. A *developable* is a surface enveloped by a plane containing only *one* variable parameter. It is generated by the intersection of two consecutive planes of the system. The class of a developable is the number of tangent planes which can be drawn to it from any point.

The locus of intersection of consecutive generators of a developable, or of three consecutive tangent planes, is called the *cuspidal edge* of the surface.

Developables and curves are closely reciprocal.

245. Quadrics. A *quadric* is a surface of the second order. Every plane section of a quadric is a conic. There are three main types of quadrics, according as they do or do not meet the plane at infinity in real points. The quadrics of the first type lie entirely at a finite distance and every section of them is an ellipse; they are called *ellipsoids*. Of the quadrics of the second type, those that do not touch the plane at infinity are called *hyperboloids*, those that do touch it *paraboloids*. Subclasses of these exist, which will be described more fully in Art. 256.

Notice that the sphere is a special case of the ellipsoid and the (real) cone of the second order a special case of the hyperboloid.

246. Tangent plane to a quadric. Consider a point P on a quadric Q. Let π be the tangent plane to Q at P. Now π meets Q in a conic s. But any straight line of π through P, being a tangent line to Q, meets Q and therefore s in two coincident points at P. s must therefore break up into a pair of straight lines, real, coincident or imaginary. They are certainly imaginary for the ellipsoid, for otherwise the points at infinity on them would be real points at infinity on Q. They are coincident if Q be a cone.

247. The two sets of generators of a quadric. We see then that a quadric Q contains an infinite number of straight lines, or generators, which lie in it. Consider one of these generators, x.

Take any point P' on the quadric, not on x. The plane $P'x$ meets the quadric in a conic, which must break up into x and another line y, which latter meets x at a point P. Thus through any point P' of the quadric there passes a generator which meets x. Draw the other generator x' through P'. x, x' cannot meet, for if they did, we should have a plane section of the quadric forming a triangle x, y, x', which is impossible. Accordingly no two generators which meet a third can themselves meet.

Hence the surface contains two sets of generators x, y such that every x meets every y and no x or y meets a generator of the same set. Also either set of generators contains every point of the surface.

The generators of a quadric, belonging to either system, are said to form a *regulus*; the regulus is reckoned, with the range and pencil of second order, as one of the elementary geometric forms.

248. Quadric as product of homographic ranges and axial pencils. If x, x' be two generators of the same system of a quadric Q, P any point on x, P' the point in which the generator y of the other system through P meets x', then clearly the correspondence between the ranges $[P]$, $[P']$ is one-one and algebraic. These ranges are therefore homographic. Hence either regulus belonging to a quadric determines homographic ranges on the lines of the other regulus. A quadric can thus be obtained as the product of two non-coplanar projective ranges.

Again let x, x' be two generators of one system, π any plane through x. π determines a generator y of the other system; y meets x'. Let the plane $yx' = \pi'$. Then the correspondence between the planes π, π' being one-one and algebraic, we have $[\pi] \overline{\wedge} [\pi']$, or a quadric can be obtained as the product of two homographic axial pencils not belonging to the same sheaf.

Conversely the product of any two such axial pencils is a quadric. For they determine on any straight line two projective ranges. These have two and only two self-corresponding points, which are the points where the product locus meets the straight line. This product locus is therefore of the second order.

Again: the product of any two non-coplanar projective ranges is a quadric. For a point of either range determines a plane through the base of the other. In this way two homographic axial pencils are formed, having the bases of the ranges for axes, and the joins of corresponding points of the original ranges are clearly the intersections of corresponding planes of the two axial pencils. These joins therefore define a quadric by the last paragraph.

The planes of either axial pencil in the above are the tangent planes at the corresponding points of the range on their own axis (since each such plane contains the other generator through its corresponding point). Hence the axial pencil formed by tangent planes to a quadric through a generator is homographic with the range formed by their points of contact on the same generator.

If we are given three non-intersecting lines x, x', x''; a variable point P on x determines two homographic axial pencils of planes $x'[P]$, $x''[P]$. The intersection of corresponding planes of these is a straight line meeting x at P and meeting also x', x'', and this is obviously the only straight line which can be drawn through P to meet x', x''. Hence:

A straight line meeting three given non-intersecting straight

lines (called *directors*) describes a regulus of a quadric, the three given lines belonging to the other regulus.

And again :

A quadric is determined by any three generators of one regulus lying in it.

249. Class of a quadric. Let a quadric be defined by the projective ranges $[P]$, $[P']$ determined by one of its reguli on two generators x, x' of the other. Let u be any straight line. The coaxial homographic axial pencils $u\,[P]$, $u\,[P']$ have two self-corresponding planes. Each one of these contains a generator PP' of the quadric. It therefore contains a second generator and touches the quadric at their intersection.

Thus through any straight line u two tangent planes can be drawn to a quadric or a quadric is a surface of the second class.

Conversely every surface of the second class is a quadric. For its section by any plane is obviously a plane curve of the second class, that is, a conic. Hence any straight line u will cut the surface of the second class in two points, for the intersections of u with the surface are its intersections with any plane section of the surface through u.

250. Twisted cubic. A *twisted cubic* is a curve of the third degree: it may be obtained as the product of three homographic axial pencils. For take any three chords a, b, c of the twisted cubic. Let P be any point on the curve and denote the planes aP, bP, cP by π_1, π_2, π_3. Now since a already meets the cubic in two points (being taken a chord), a plane π_1 through it can meet the cubic again in one point P only. Thus when π_1 is given, P, and therefore π_2 and π_3, are uniquely determined. Similarly if π_2 or π_3 be given, the other two are uniquely determined. Hence $[\pi_1]$, $[\pi_2]$, $[\pi_3]$ are three homographic axial pencils of planes, of which the twisted cubic is the product.

The intersection of two quadrics having a common generator a is a twisted cubic. For the two quadrics meet any plane in two conics. Their curve of intersection therefore meets the plane in the four points of intersection of these conics. In general therefore the intersection of two quadrics is a twisted quartic. But if they have a common generator, this generator accounts for one of these four intersections. The remaining part of the intersection of the two quadrics then meets any plane in three points, that is, it is a twisted cubic.

It follows that *any* three homographic axial pencils of planes $[\pi_1]$, $[\pi_2]$, $[\pi_3]$ determine a twisted cubic as their product. For the product of $[\pi_1]$ and $[\pi_2]$ is a quadric Q_2 having the axes of $[\pi_1]$, $[\pi_2]$ as generators of the same system; and the product of $[\pi_1]$ and $[\pi_3]$ is a quadric Q_3 having the axes of $[\pi_1]$ and $[\pi_3]$ as generators of the same system. Q_2 and Q_3 have a common generator, namely the axis of $[\pi_1]$. The remainder of their intersection is a twisted cubic, which is the product of the three given axial pencils.

In particular cases this twisted cubic will degenerate into a straight line and a conic, or into three straight lines. For example if the axial pencils $[\pi_1]$, $[\pi_3]$ have a self-corresponding plane α, Q_3 breaks up into this plane α and another β by Art. 230. The intersection of Q_2, Q_3 then consists of two straight lines in α (one of which is the axis of π_1) and a conic in β.

Again if the axes of the three pencils $[\pi_1]$, $[\pi_2]$, $[\pi_3]$ all lie in one plane α and this plane be self-corresponding for any pair of pencils, Q_2 and Q_3 break up into planes α, β and α, γ respectively. The twisted cubic then reduces the straight line $\beta\gamma$, together with two *indeterminate* straight lines in the plane α, for it is easy to see that $\alpha\beta$, $\alpha\gamma$ do not in this case give a definite locus since a *third* straight line in α, of the same type, can be got by pairing the pencils differently.

If two quadrics have two generators x, x' of one system in common, the remainder of their intersection consists of two generators of the other system. For let P be a point on the remainder of their intersection. Through P a line y can be drawn to meet x and x' at Q and R. P, Q, R are three points on each quadric: the line y which contains them is thus a generator of each quadric, since a straight line which meets a quadric at three points must lie entirely in the quadric.

251. Intersections of three quadrics. By Art. 243 three quadrics Q_1, Q_2, Q_3 will intersect in general in eight points.

A twisted cubic t meets any quadric Q_1 in six points. For such a cubic is obtainable as the product of three homographic axial pencils, that is, as the intersection of the two quadrics Q_2, Q_3 determined by two pairs of these pencils and having a common generator x.

Of the eight intersections of the three quadrics Q_1, Q_2, Q_3 two are accounted for by the intersections of Q_1 with x. The remaining six are the intersections of Q_1 with the twisted cubic.

A twisted cubic t which is the intersection of two quadrics Q_1

and Q_2 which have a common generator x, will intersect x at two points.

For any plane through x meets each quadric in a straight line. The intersection of these two lines is the only point of the plane not on x which is common to the quadrics. Hence the remaining two points A, B in which the cubic intersects this plane lie on x. The tangent line at A to the cubic and x form two lines through A which are tangent to both quadrics at A. The two quadrics therefore touch at A and similarly they touch at B.

It follows that three quadrics Q_1, Q_2, Q_3 which have a common generator x will intersect in four other points outside x. For the twisted cubic determined by Q_2, Q_3 intersects Q_1 in six points. But two of these points are clearly those where the twisted cubic meets x. There are thus only four intersections left.

252. Pole and polar plane. Let P be any point and let any ray through P meet the quadric at R, S. If P' be the point harmonically conjugate to P with regard to R, S, then P' lies on a fixed plane.

For take two rays through P, PQ_1R_1, PQ_2R_2 and let $P_1'P_2'$ be the corresponding positions of P'. Join $P_1'P_2'$. Then if α be the plane of the two rays PQ_1R_1, PQ_2R_2, $P_1'P_2'$ is the polar of P with regard to the conic in which α meets the quadric. Hence the locus of points P' corresponding to all rays through P which lie in α is the straight line $P_1'P_2'$. Thus the straight line joining any two points on the locus lies entirely in the locus. But this property defines a plane.

This plane is called the *polar plane* of P with regard to the quadric.

When the points R, S coincide, P' coincides with them. Thus the tangent cone from P touches the quadric along a plane section. This cone is therefore of the second order.

If P lie on the polar plane ρ of R, and PR meet the quadric at (S, T), then (P, R) are harmonic conjugates with regard to S, T and therefore the polar plane π of P passes through R.

P, R are conjugate points and π, ρ conjugate planes with regard to the quadric.

Consider the poles of planes through P. These lie on its polar plane π. Similarly the poles of planes through R lie on ρ. Thus the poles of planes through PR lie on a fixed line $\pi\rho$.

Hence if S be any point of PR, S' any point of $\pi\rho$, SS' is harmonically divided by the quadric.

The symmetry of this last relation shows that the poles of planes through $\pi\rho$ lie on PR.

Two such lines PR, $\pi\rho$ are said to be conjugate or polar lines with regard to the quadric.

Two conjugate planes π, ρ divide harmonically the dihedral angle between the two tangent planes σ, τ through their intersection $\pi\rho$. For let S, T be the points of contact of σ, τ and let ST meet π, ρ at P, R respectively. Then if U be any point of $\pi\rho$ the polar plane of U passes through ST. $\pi\rho$, ST are therefore conjugate lines. Thus the pole of ρ lies on ST; but it also lies on π, hence it must be P; $\therefore \{PSRT\} = -1$, or $\{\pi\rho\sigma\tau\} = -1$, which proves the required result.

The polar plane of a point P on the quadric is the tangent plane at P. For the polar plane of every point R in the tangent plane at P passes through P.

253. Self-polar tetrahedron. Let P be any point, π its polar plane, R a point of π, ρ its polar plane, which passes through P, S a point of $\pi\rho$, σ its polar plane, which passes through P, R and meets $\pi\rho$ at T. Then T lies on π, ρ, σ: its polar plane τ is PRS. A tetrahedron such as $PRST$ is said to be self-conjugate or self-polar with regard to the quadric. Each vertex is the pole of the opposite face. Any two of its vertices, or any two of its faces, or any two of its opposite edges, are conjugate with regard to the quadric.

The three-edge formed by any three faces of a self-polar tetrahedron is said to be a self-polar three-edge for the quadric. Any two of the faces are conjugate and the pole of any face lies in the opposite edge. The polar plane of the vertex of such a three-edge meets it in a triangle self-polar for the conic in which the same plane meets the quadric. This three-edge is therefore also (Art. 231) self-polar with regard to the tangent cone to the quadric from its vertex.

254. Asymptotic cone. The tangent planes at the points of intersection of a plane π with the quadric pass through the pole of π. Taking π to be the plane at infinity, the tangent planes at infinity to the quadric envelop a cone of the second order which is called the asymptotic cone of the quadric. Its vertex is the pole of the plane at infinity. This is called the centre of the quadric. Lines and planes through the centre are diameters and diametral planes respectively. As in the case of the conic all diameters are bisected at the centre.

A self-polar three-edge whose vertex is the centre forms a system of three conjugate diametral planes. Any one of its edges is said to be a diameter conjugate to the opposite diametral plane, although strictly speaking the conjugate to a diameter is the line at infinity of its so-called conjugate diametral plane. The pole of a diametral plane is the point at infinity on the conjugate diameter. Hence chords parallel to a diameter are bisected by its conjugate diametral plane.

Also if C be the point where a diameter meets a plane γ' parallel to its conjugate diametral plane γ, C is conjugate to the points of the line at infinity of γ or γ' and is the centre of the section of the quadric by γ'.

A set of three conjugate diametral planes forming also by Art. 253 a self-polar three-edge for the asymptotic cone, properties of such three-edges follow immediately from the theory of the cone of second order in Chap. XIII. Among other results we see that there exists one set and one only of three real mutually perpendicular conjugate diametral planes. These are called the *principal planes* of the quadric. They are planes of symmetry both for the quadric and its asymptotic cone. Their intersections are called the *axes* of the quadric.

255. Planes of circular section. Since the quadric and its asymptotic cone have their points at infinity common, it follows by Art. 228 that the sections of the quadric and the cone by any plane have the same points at infinity and are therefore similar. In particular, if one of these sections is a circle, so is the other. Hence planes parallel to the cyclic planes of the asymptotic cone cut the quadric in circles. The tangent planes parallel to the planes of circular section meet the quadric in point-circles. Their points of contact are called *umbilics* of the quadric. Since there are six cyclic planes, of which two are real, there are twelve umbilics, of which four are real; and they lie in fours in the three principal planes.

Also if the asymptotic cone is right circular, the quadric is a surface of revolution whose axis is the axis of its asymptotic cone. In this case the section of either the asymptotic cone or the quadric by the plane at infinity has double contact with the circle at infinity (Art. 237). Conversely if a quadric has double contact with the circle at infinity it is a surface of revolution.

If the asymptotic cone be a spherical cone the quadric is a sphere and every set of conjugate diametral planes are rectangular.

256. Classification of hyperboloids and paraboloids. The hyperboloid, since it meets the plane at infinity in a real conic, has a real asymptotic cone. There are two classes of hyperboloids, according as the surface lies inside, or outside, its asymptotic cone. In the first case there are two sheets to the surface, one lying inside each half cone, and since no straight line not passing through the vertex can lie entirely inside a cone of the second order, there can be no real generators of such a quadric. It is called a *hyperboloid of two sheets.* In the second case there is only one sheet to the surface. Also a tangent plane at infinity to the surface, being also a tangent plane to the asymptotic cone, has points lying inside the surface and so cuts the latter in real lines. This quadric has therefore real generators. It is called a *hyperboloid of one sheet.*

Since the paraboloid touches the plane at infinity, it meets this plane in two straight lines. The centre is here the point of contact of the plane at infinity and is itself at infinity. The tangent planes to the asymptotic cone become two sets of parallel planes through the generators at infinity. Also in the ranges determined by one regulus upon the other, since each regulus contains a ray at infinity, the ranges have their points at infinity corresponding and so are similar. Conversely joins of corresponding points of two similar ranges generate a paraboloid.

There are two classes of paraboloids according as they meet the plane at infinity in two real or in two imaginary lines. The first class are called *hyperbolic paraboloids.* Any tangent plane must contain two real points at infinity on the quadric and so meets the latter in real generators.

The second class are called *elliptic paraboloids.* Every plane section of these has imaginary points at infinity and so is an ellipse, except sections by planes parallel to the direction of the actual point of contact of the quadric with the plane at infinity; these meet the quadric in parabolas. It is obvious that such a quadric can have no real generators.

257. Reciprocal polars with regard to a quadric. As in the case of plane figures, so in space, we can construct a reciprocal transformation in which to each point corresponds its polar plane with regard to a quadric Q called the base quadric, and conversely. To points of a plane will correspond planes through a point: to a plane figure or a sheaf will correspond a sheaf or plane figure respectively, and to ranges, flat pencils, and axial pencils will correspond homographic axial pencils, flat pencils,

and ranges respectively. To a surface of the second order will correspond one of the second class and conversely, that is, a quadric corresponds to a quadric.

258. Pencil of quadrics. The equation of a surface of second order contains *ten* coefficients, the *nine* ratios of which determine the equation. A quadric is therefore, in general, determined by nine points.

If eight points are given on a quadric, the coefficients of its equation satisfy eight linear relations. The ten coefficients can therefore be expressed as homogeneous linear functions of two arbitrary parameters λ_1, λ_2. The equation of the quadric then takes the form

$$\lambda_1 S_1 + \lambda_2 S_2 = 0,$$

where S_1, S_2 are expressions of the second degree in the coordinates. This represents a quadric passing through the intersection of the quadrics

$$S_1 = 0, \quad S_2 = 0.$$

Hence the set of quadrics through eight points contains a given twisted quartic.

Such a set of quadrics are said to form a pencil of quadrics.

259. Range of quadrics. Reciprocating the above results we see that the quadrics which touch eight given planes touch a given developable of the fourth class. Such a set form a range of quadrics.

260. Properties of a pencil of quadrics. A pencil of quadrics determines an involution on any straight line. Two quadrics of the pencil touch this straight line at the two double points of this involution.

The conics in which the quadrics of a pencil meet any plane form a pencil of conics passing through the four points in which the twisted quartic which defines the pencil meets this plane. Three of the quadrics of the pencil therefore meet the plane in line-pairs, that is, they touch the plane at the centres of the line-pairs. These centres are the vertices of the common self-polar triangle of the conics in which the pencil of quadrics meets the plane. Hence:

In every plane there is one triangle self-polar with regard to all the quadrics of a pencil. Its vertices are the points of contact of the three quadrics of the pencil which touch the plane.

261. Self-polar tetrahedron of a pencil of quadrics. Let Q_1, Q_2 be two quadrics, P any point. The points of space which are conjugate to P with regard to both quadrics are on the intersection p of the polar planes π_1 and π_2 of P with regard to Q_1, Q_2.

Also if two points P, P' are conjugate with regard to both Q_1 and Q_2 they are clearly conjugate with regard to every quadric through the intersection of Q_1 and Q_2, for, being harmonically conjugate with regard to two pairs of mates of the involution determined on PP' by the quadrics of the system, they are the double points of this involution and so harmonically conjugate with regard to every such pair of mates; therefore they are conjugate for every quadric of the pencil.

Thus to every point P of space corresponds a line p every point of which is conjugate to P with regard to all the quadrics of the pencil. We may call this the line conjugate to P with regard to the pencil.

If P describe a straight line u, the planes π_1, π_2 sweep out two homographic axial pencils about the polar lines u_1, u_2 of u with regard to Q_1, Q_2. Thus p generates a quadric U which is the locus of points conjugate to points of u for the pencil.

This quadric U contains u_1 and u_2. By symmetry it must contain the polar line of u with regard to every conic of the pencil. Thus the polar lines of u and the lines conjugate to points of u form the two systems of generators of U.

Let u, v, w be three lines through a point A, a the line conjugate to A for the pencil. The corresponding quadrics U, V, W have a common generator a. They have thus four common points P, R, S, T (Art. 251). There are three points P', P'', P''' on u, v, w respectively, which are conjugate to P for the pencil. The plane $P'P''P'''$ is therefore the polar plane of P for all quadrics of the pencil. Conversely a point which has the same polar plane with regard to all quadrics of a pencil is necessarily an intersection of U, V, W. There are then only *four* such points. Now the polar planes of P, R, S meet at the pole of the plane PRS for all quadrics of the pencil. This pole must accordingly be T. Similarly P, R, S are the poles of RST, STP, TPR respectively. Hence $PRST$ is the common self-polar tetrahedron for all quadrics of the pencil.

262. Cones through the intersection of two quadrics. Let P be one of the vertices of the common self-polar tetrahedron of the quadrics, A a point of the twisted quartic

in which they intersect. Let PA meet the polar plane of P at L. Then if A' be harmonically conjugate to A with regard to P and L, A' lies on both quadrics and therefore on the quartic.

Hence the four points A, B, A', B' in which any plane through P meets the quartic lie in two pairs $(AA')(BB')$ on rays through P. The lines joining P to the quartic form a cone which has two generators in any plane through P, that is, a cone of the second order. A similar result holds for the other vertices R, S, T of the common self-polar tetrahedron. Hence:

Four of the quadrics of a pencil are cones, whose vertices are the vertices of the common self-polar tetrahedron of the pencil.

263. **Properties of a range of quadrics.** The properties of a range of quadrics are immediately derivable from those of a pencil of quadrics by reciprocation. We will note the following:

The tangent cones from any point to the quadrics of a range form a system touching four planes.

Through any point three quadrics of the range can be made to pass.

To any plane π corresponds a line p through which pass all planes conjugate to π for the quadrics of the range and which is also the locus of poles of π for the quadrics of the range.

Taking π at infinity the locus of centres of the quadrics of a range is a straight line.

The surface generated by the lines p corresponding to planes π through a given point P is a quadric touching four fixed planes independent of the position of P. These four fixed planes are the faces of a tetrahedron self-polar for all the quadrics of the range.

Bearing in mind that a cone reciprocates into a conic (which is therefore to be considered as a special case of a surface of the second class) we see, reciprocating the property of the last Article, that:

Four of the quadrics of a range are conics, whose planes are the faces of the common self-polar tetrahedron of the range.

264. **Confocal quadrics.** Consider the range determined by any quadric Q and the circle at infinity (a degenerate case of a quadric). There are three conics of this range, besides the circle at infinity. Their planes α, β, γ and the plane at infinity form the self-polar tetrahedron of the range: α, β, γ are therefore three conjugate diametral planes of any quadric Q' of the range. They form a self-polar three-edge of the asymptotic

cone of Q'. But also they must meet the plane at infinity in a self-polar triangle for the circle at infinity. Hence they form a self-polar three-edge of a spherical cone, that is, a rectangular three-edge. Or α, β, γ are the three principal planes of Q'.

Hence the quadrics Q' of such a range are concentric and coaxial. There are three conics of the range lying each in one of the three common principal planes.

These conics are called the *focal conics* of Q' : every point of them is called a *focus* of Q'.

The quadrics Q' are said to form a *confocal* system.

Let F be any point of a focal conic. Then the tangent cones from F to the quadrics of the confocal system form by Art. 263 a system of cones touching four fixed planes through F. Now consider the tangent cone to a conic from any point in its plane. This tangent cone (treated as an envelope) reduces to the two tangents from the point to the conic. Hence the tangent cone from F to the focal conic consists of two coincident tangents to this conic at F. The four fixed planes therefore consist of the two tangent planes to any cone of the system through the tangent line to the focal conic at F, each such tangent plane being *doubled*, that is, its line of contact being given. Hence every cone of the system touches two fixed planes through F along given lines through F in these planes, or the tangent cones from F to the system of confocals have double contact. But one of these tangent cones is the tangent cone to the circle at infinity, that is, it is the spherical cone through F. The tangent cones from F to the system of confocals have therefore double contact with the spherical cone ; that is, they are right circular cones.

Foci of a quadric are thus points, the tangent cones from which to the quadric are right circular.

265. Net of quadrics. If a quadric passes through seven given points, we can show as in Art. 258 that its equation may be put into the form

$$\lambda_1 S_1 + \lambda_2 S_2 + \lambda_3 S_3 = 0,$$

S_1, S_2, S_3 being given expressions of the second degree in the coordinates and λ_1, λ_2, λ_3 arbitrary parameters. This quadric passes through the intersections of the three quadrics

$$S_1 = 0, \quad S_2 = 0, \quad S_3 = 0,$$

that is, quadrics satisfying such a condition pass through eight fixed points. Thus, in addition to the seven given points, there is an eighth fixed point, which is determined by the seven first, and through which the quadrics pass.

The quadrics through seven given points are said to form a *net* of quadrics.

The quadrics of a net which pass through another given point P_1 form a pencil, and so have a twisted quartic in common.

Consider then two pairs of quadrics of the net (Q_1, Q_1') and (Q_2, Q_2'). Let P_1 be a point on the intersection of (Q_1, Q_1') other than the seven given points, or the eighth point which depends on them. Then the quadrics of the net, which pass through P_1, contain the intersection of Q_1 and Q_1'. Similarly if P_2 be a point on the intersection of Q_2 and Q_2' the quadrics of the net which pass through P_2 contain the intersection of Q_2 and Q_2'. Therefore the quadric of the net which passes through both P_1 and P_2 contains the intersections of Q_1, Q_1' and of Q_2, Q_2'.

We deduce that if four quadrics Q_1, Q_2, Q_3, Q_4 be such that the intersection of Q_1, Q_2 and that of Q_3, Q_4 lie on a quadric Q the same is true however we choose the two pairs out of the four quadrics.

For consider the net defined by the quadrics Q_1, Q_2, Q_3. Any quadric through the intersection of two quadrics of the net is a quadric of the net. Therefore Q is a quadric of the net; therefore Q_4 which passes through the intersection of Q_3 and Q is a quadric of the net. Q_1, Q_2, Q_3, Q_4 are therefore four quadrics of a net and the result follows.

We obtain also the following important theorem of plane geometry.

If there be four conics s_1, s_2, s_3, s_4 such that the four points of intersection of s_1, s_2 and the four points of intersection of s_3, s_4 lie on a conic s, the same is true of any other two pairs chosen out of the four conics.

Through s_1, s_2, s_3 describe any three quadrics Q_1, Q_2, Q_3. These will define a net. A quadric Q of the net can be drawn through one of the intersections of s_1, s_2 and one of the intersections of s_3, s_4. It will therefore contain, besides the eight points common to Q_1, Q_2, Q_3, another point common to Q_1, Q_2 and thus the whole intersection of Q_1, Q_2. Hence Q contains the four intersections of s_1, s_2 and one intersection of s_3, s_4; therefore it contains the conic s. Now through the intersection of Q and Q_3 draw a quadric Q_4 to pass through any given point of s_4. This quadric cuts the plane in a conic having five points common with s_4 and therefore identical with s_4.

Four such conics s_1, s_2, s_3, s_4 are therefore the intersections

of four quadrics of a net by a plane. The theorem is then obvious.

266. Conjugate points with regard to a net of quadrics. If two points P, P' are conjugate with regard to three quadrics Q_1, Q_2, Q_3 of a net, they are conjugate with regard to all quadrics of the net. For if Q_4 be any other quadric of the net, we have seen by the above that a quadric Q exists belonging to both pencils (Q_1, Q_2) and (Q_3, Q_4). If P, P' are conjugate with regard to (Q_1, Q_2) they are also (by Art. 261) conjugate with regard to Q. Also Q, Q_3, Q_4 are quadrics of a pencil. Hence P, P' being conjugate with regard to Q, Q_3, they are also conjugate with regard to Q_4.

Thus to every point P of space corresponds a point P' conjugate to P with regard to the net. P' is obtained as the intersection of the polar planes of P with regard to any three quadrics of the net. Hence the polar planes of P with regard to the quadrics of the net pass through a fixed point P'.

267. Webs of quadrics. A *web* of quadrics is the set of quadrics touching seven fixed planes. Reciprocating the properties of a net of quadrics we obtain the following:

The quadrics of a web touch an eighth fixed plane. If four quadrics Q_1, Q_2, Q_3, Q_4 belong to a web the common tangent planes to Q_1 and Q_2 and the common tangent planes to Q_3 and Q_4 all touch a fixed quadric Q.

To every plane of space there is one plane conjugate with regard to a web of quadrics, i.e. the poles of a fixed plane with regard to the quadrics of a web lie on another fixed plane.

In particular if the given plane be taken at infinity the locus of centres of quadrics of a web is a plane.

EXAMPLES XIV.

1. The locus of the vertex of a cone of the second order inscribed in a given skew hexagon is a quadric.

2. A regulus lying in a quadric and an axial pencil homographic with the regulus generate a twisted cubic.

3. If a sheaf of tangent planes to a cone of the second order be homographic with an axial pencil not passing through the vertex of the cone, but such that the sheaf and the axial pencil have one self-corresponding plane, their product is a quadric.

4. A range of points on a conic is homographic with a range on a straight line not coplanar with the conic but meeting the conic at A. If A be a self-corresponding point show that the joins of corresponding points of the two ranges lie on a quadric.

5. Two fixed straight lines a and b meet a conic s, but are not coplanar with s or with each other. Show that a straight line which meets s, a, b describes a quadric.

6. A tangent plane to the asymptotic cone meets the quadric in parallel generators belonging to opposite systems.

7. Through the points where the planes of an axial pencil meet a straight line are drawn perpendiculars to these planes. Show that these perpendiculars lie in a hyperbolic paraboloid.

8. Show that if a quadric contain a twisted cubic, the generators of one set meet the cubic in one point, while those of the other set meet it in two points.

9. Show that if two quadrics have a common generator the generators of the other system in each quadric, which intersect on their common twisted cubic, form homographic reguli.

10. Prove that through any point P of space a quadric can be drawn containing a given twisted cubic and a given chord of it.

Show that through P one chord, and one only, of a given twisted cubic can be drawn.

11. Show that a regulus projects from any point upon any plane into a homographic pencil of the second order.

12. α, β are two planes; a, b two non-coplanar lines in space which both meet $\alpha\beta$. Show that if P_1, P_2 be points of α, β respectively such that P_1P_2 meets a and b, the correspondence between the planes $[P_1]$, $[P_2]$ is homographic.

13. Prove that the locus of the poles of a fixed plane for the quadrics of a pencil is a twisted cubic.

[It is the intersection of the quadrics of conjugate points for two lines in the fixed plane.]

Deduce that the locus of centres of quadrics of a pencil is a twisted cubic whose asymptotes are parallel to the directions of the points of contact with the plane at infinity of the paraboloids of the system.

14. If two quadrics of a pencil meet a plane in conics harmonically circumscribed to the same conic s, show that this is true of all the quadrics of the pencil.

By taking s to be the circle at infinity, show that if two quadrics of a pencil are equilateral (i.e. are such that rectangular three-edges can be inscribed in their asymptotic cone) then all quadrics of the pencil possess the same property.

15. Show that a single focal conic defines a family of confocal quadrics and that through any point of space three quadrics of the family can be drawn, which cut orthogonally.

16. Prove that the poles of a fixed plane with regard to a system of confocal quadrics lie on a fixed line normal to the plane.

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THE END

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